MATH 155R: ALGEBRAIC COMBINATORICS

COLIN DEFANT

1. Tuesday September 5

CAs: Katherine Tung and Eliot Hodges

1.1. **Bookkeeping.**

(1) Colin's Office Hours: Thursdays from 3-4pm in SC 235 (Colin's office). The time may be subject to change based on the availability of the students in the class.
(2) CA Office Hours: TBD. Keep an eye out for a when2meet on Canvas. CA office hours will be scheduled based on your availability.
(3) Grading policy: There are 5 problem sets. Homework is worth 80% of the grade, with a final exam making up the last 20%. Each student gets 5 late days to be used at their discretion throughout the semester. See the syllabus for a more detailed explanation of grading policies.

1.2. **Course Intro.** The symmetric group S_n is the group of permutations of the first n integers, which we denote by $[n] := \{1, \ldots, n\}$. The group operation is composition. A permutation can be represented in *one-line notation* or *cycle notation.*

Example 1.1. Consider S_6 and the permutation taking $1 \mapsto 4$, $2 \mapsto 6$, $3 \mapsto 5$, $4 \mapsto 3$, $5 \mapsto 1$, and $6 \mapsto 2$. In one-line notation, we represent this element by 465312. In cycle notation, we represent it by $(1435)(26)$.

Note that we can immediately see that $|S_n| = n!$.

An *inversion* of $w \in S_n$ is a pair (i, j) where $i < j$ and $w^{-1}(i) > w^{-1}(j)$. In one-line notation, (i, j) with $i < j$ is an inversion if j appears before i.

Example 1.2. Consider $31425 \in S_5$. The inversions are $(1, 3)$, $(2, 3)$, and $(2, 4)$.

Here's another fact:

$$\sum_{w \in S_n} q^{\mathrm{inv}(w)} = \prod_{i=1}^{n} \frac{1 - q^i}{1 - q},$$

where inv(w) denotes the number of inversions of w and q denotes a formal variable.

Say $\tau \in S_k$. Say a permutation $w \in S_n$ *contains* τ if there is a subsequence τ if there is a subsequence of w that has the same relative order as τ. We say w *avoids* τ if it does not contain τ.

Example 1.3. 421635 contains 231 (look at the 4, 6, and 3 in the first permutation). As an exercise, check for yourself that 312645 avoids 231.

Fact: Fix some $\tau \in S_3$. The number of permutations in S_n that avoid τ is $C_n = \frac{1}{n+1}\binom{2n}{n}$. The numbers C_n are the *Catalan numbers*, a very important sequence of numbers in algebraic combinatorics. The first few of these are: $1, 2, 5, 14, 42, 132, \ldots$.

Question 1.4. (Open) How many permutations in S_n avoid 1324? Although this question is easy to state, this is actually very hard.

Exercise 1.5. There are several facts in the lecture that were stated without proof. If you haven't seen these before (or even if you have), try proving them yourself!

A *descent* of a permutation w of S_n is an index $i \in [n-1]$ such that $w(i) > w(i+1)$. A *transposition* is a 2-cycle, i.e., a permutation that swaps two numbers and fixes everything else. Write (ij) for the transposition swapping i and j.

1.3. **S_n as a Coxeter Group.** Let $s_i = (i\ i+1) \in S_n$ for $i \in [n-1]$. The s_i's are called *adjacent transpositions*, *simple transpositions*, or *simple reflections*. The following is a good fact:

Fact 1.6. The symmetric group S_n is generated by $s_1, \ldots, s_{n-1}$. The s_i's satisfy the following relations:

(1) $s_i^2 = e$ (where e denotes the identity element);
(2) $(s_i s_j)^2 = e$ if $|i-j| \geq 2$;
(3) $(s_i s_{i+1})^3 = e$.

The above set of relations is equivalent to the following set of relations:

(1) $s_i^2 = e$;
(2) $s_i s_j = s_j s_i$ if $|i-j| \geq 2$ (*commutation relations*);
(3) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (*braid relations*).

The s_i's with either of the above sets of relations form a *presentation* of S_n.

Example 1.7. We illustrate Fact 1.6 with an example. Consider $w = 52341 \in S_5$. We can write w as a product of simples by multiplying on the right by simples until we get the identity on the left:

$$12345 = w s_1 s_4 s_2 s_3 s_4 s_2 s_1.$$

Hence, $w = s_1 s_2 s_4 s_3 s_2 s_4 s_1$. However, this way of writing w as a product of simples is not unique (consider $s_1 s_2 s_4 s_3 s_2 s_4 s_1 s_3 s_3$).

A *reduced word* for a permutation w is a way of writing w as product of simples such that the product has minimal length (see the product of simples in the example above—is $s_1 s_2 s_4 s_3 s_2 s_4 s_1$ a reduced word for 52341?).

Fact 1.8. The number of ways to write the permutation $n(n-1)\cdots 321$ is

$$\frac{\binom{n+1}{2}!}{1^n 3^{n-1} 5^{n-2} \cdots (2n-1)^1}.$$

1.4. **Coxeter Groups.** We would like to generalize the objects (e.g., simple transpositions, reduced words, etc.) introduced for S_n in the previous subsection to *Coxeter groups*. Coxeter groups bring together the following fields of study:

(1) Combinatorics
  - generalize from S_n
  - reduced words
  - posets (Bruhat order, weak order, absolute order, convex sets)
(2) Geometry
  - Hyperplane arrangements
  - polytopes
  - root systems

(3) Algebra
- Group theory (duh)
- Weyl groups of semisimple Lie algebras
- Representation theory, Hecke algebras, etc.

N.B. The class is not exclusively about Coxeter groups, though they indeed are our main object of study. Throughout we will see other objects of interest from algebraic combinatorics.

Definition 1.9. Let S be a (usually finite) set. For $s, s' \in S$, choose $m(s, s') = m(s', s) \in \mathbb{N} \cup \{\infty\}$ such that $m(s, s) = 1$ for all $s \in S$ and $m(s, s') \geq 2$ if $s \neq s'$. This will be the data that defines our Coxeter group. Let W be the group with presentation

$$W = \langle S \mid (ss')^{m(s,s')} = e \text{ for all } s, s' \in S \rangle.$$

The pair (W, S) is called a *Coxeter system*; S is called the set of *simple generators* (or *simple reflections*); W is called a *Coxeter group*.

Remark 1.10. For all $s \in S$, we have $s^2 = e$. If $m(s, s') = 2$, then s and s' commute: $ss'ss' = e$ implies $s's = ss'$. If $m(s, s') \geq 3$, then s and s' do not commute.

There is a nice way of representing the data of a Coxeter system using a graph called the *Coxeter graph* (sometimes *Dynkin diagram*).

Definition 1.11. The *Coxeter graph* of (W, S) is the graph with vertex set S, where s and s' are adjacent when $m(s, s') \geq 3$. If $m(s, s') \geq 4$, we label the edge between s and s' with this number.

Example 1.12. Picture I can't draw quickly in tikz. In words, $S = \{s_1, s_2, s_3\}$; there are edges between s_i and s_{i+1} for $1 \leq i \leq 2$, where the edge between s_1 and s_2 is labeled by 4 and the edge between s_2 and s_3 is unlabeled. Quickly, we see that $m(s_1, s_2) = 4$, $m(s_2, s_3) = 3$, and $m(s_1, s_3) = 2$.

Example 1.13. Consider $S = \{s_1, \ldots, s_n\}$ with no edges between the vertices. Here we see that $W \simeq (\mathbb{Z}_2)^n$.

Example 1.14. Consider $S = \{s_1, \ldots, s_n\}$ with unlabeled edges between s_i and s_{i+1} for $i \in [n-1]$. Then $W \simeq S_n$. Note that $m(s_i, s_i) = 1$, $m(s_i, s_j) = 2$ if $|i - j| \geq 2$, and $m(s_i, s_{i+1}) = 3$. As an exercise, prove that $W \simeq S_n$.

Example 1.15. Consider $S = \{s_1, s_2, s_3\}$ with an unlabeled edge between s_1 and s_2 and no other edges. Then $W \simeq S_3 \times \mathbb{Z}_2$. Note that if the Coxeter graph is disconnected, then W is the direct product of the groups given by each connected component of the graph.

Example 1.16. Consider $S = \{r, s\}$ with an edge between r and s labeled by m. Then $W = D_m$, the *dihedral group* of order $2m$. There is a nice picture here that I can't draw.

2. Thursday September 7

2.1. Last Time.

Recall that (W, S) is a Coxeter system, where S is the set of *simple generators* (or *simple reflections*) and W is the Coxeter group. We have the following presentation of W, where

$$W = \langle S \mid (ss')^{m(ss')} = e \rangle$$

and

(1) $m(s, s) = 1$;
(2) $m(s, s') = m(s', s) \in \{2, 3, \ldots\} \cup \{\infty\}$ if $s \neq s'$.

Proposition 2.1 (Proposition 1.1.1 in Bj orner-Brenti). *For $s, s' \in S$, the order of ss' is $m(s, s')$.*

We'll postpone the proof until later on in the course.

A Plethora of Examples. Here are some more examples of Coxeter groups.

Example 2.2. Consider $S = \{s_0, \dots, s_{n-1}\}$ where there are edges between s_i and s_{i+1} for $i \in \{0, \dots, n-2\}$. Label the edge between s_0 and s_1 by 4; the rest of the edges are unlabeled. This is called the nth *hyperoctahedral group* and is denoted B_n. Alternatively, B_n can be viewed as the group of permutations of $\{-n, \dots, -1, 1, \dots, n\}$ such that $w(-i) = -w(i)$ for all i and $w \in B_n$. Note that B_n is isomorphic to a subgroup of S_{2n}.

Elements of B_n are called *signed permutations*. We can represent these elements using a one-line notation. We'll do definition by example: $\bar{2}14\bar{3} \in B_4$ denotes the permutation taking $1 \mapsto -2$, $2 \mapsto 1$, $3 \mapsto 4$, and $4 \mapsto -3$; from here we can determine that $-1 \mapsto 2$, $-2 \mapsto -1$, $-3 \mapsto -4$, and $-4 \mapsto 3$. In other words, it's enough to specify where $1, \dots, n$ go; $\bar{i}$ denotes $-i$ in the one-line notation.

Now, what are the simple reflections? For $1 \le i \le n-1$, we have $s_i = (i\ i+1)(-i\ \ -(i+1))$ and $s_0 = (-1\ 1)$.

Finally, as an exercise, show that $|B_n| = 2^n n!$ (hint: count the elements using the one-line notation). See Homework 1 to learn about the Coxeter group D_n.

Example 2.3. The *affine symmetric group*, denoted $\widetilde{S_n}$, is the group of permutations w of $\mathbb{Z}$ such that $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$ and $w(1) + w(2) + \cdots + w(n) = \binom{n+1}{2}$. For example, consider $n = 4$ and the permutation given by

$$s_0 = \begin{pmatrix} -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ -3 & -4 & -2 & -1 & 1 & 0 & 2 & 3 & 5 & 4 \end{pmatrix}.^{1}$$

In general, we let

$$s_i = \cdots (i-n\ \ i-n+1)(i+n\ \ i+n+1)(i+2n\ \ i+2n+1) \cdots$$

be the $i+1$st simple reflection. The Coxeter graph is a cycle of size n. where s_i is adjacent to s_{i+1} for all i, where the indices are taken modulo n. Finally, note that $|\widetilde{S_n}| = \infty$.

Example 2.4. The symmetry group of a regular polytope is a Coxeter group. Consider the following interesting examples.[2]

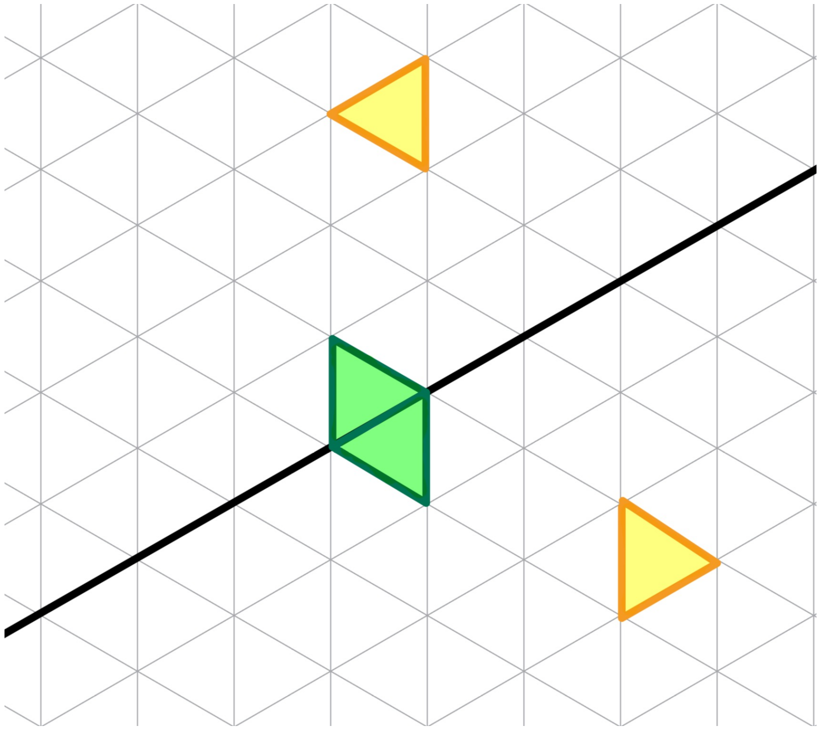
Dimension	Regular Polytope	Coxeter Graph
d	simplex	$A_d \simeq S_{d+1}$
d	cube	B_d
d	hyperoctahedron	B_d
2	m-gon	$I_2(m)$
3	dodecahedron	H_3
3	icosahedron	H_3
4	24-cell	F_4
4	120-cell	H_4
4	600-cell	H_4

Coxeter groups can also be realized as reflection groups; we can realize this via the following example.

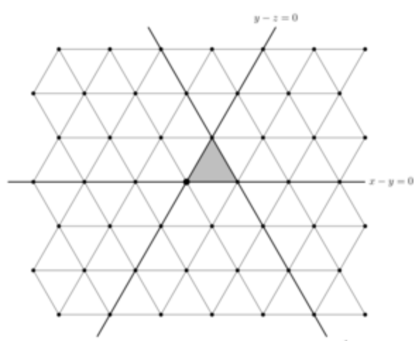
Example 2.5. Consider $\widetilde{S_3}$.

[1] See if you can find the values of this permutation for n of larger absolute value.

[2] I can't draw graphs for these (sorry!), but I'll refer you to the Wikipedia page on Coxeter groups which will certainly have pictures.



For each line in the above picture, we can reflect across this line; $\widetilde{S}_3$ is the group generated by all such reflections. In the picture below, the three bolded lines are the three simple reflections. As an exercise, justify to yourself that any of the reflections can be written as a composition of the three simple reflections.



In some sense, *any* Coxeter group can be regarded as the reflection group of some space (though sometimes the examples will be weird and we'll be reflecting through lines in hyperbolic space).

2.2. **Irreducible Coxeter Groups.**

Definition 2.6. A Coxeter group W is said to be *irreducible* if its Coxeter graph is connected. Otherwise, W is said to be *reducible.* As an exercise, prove that a reducible Coxeter group is the direct product of the Coxeter groups given by the connected components of the graph.

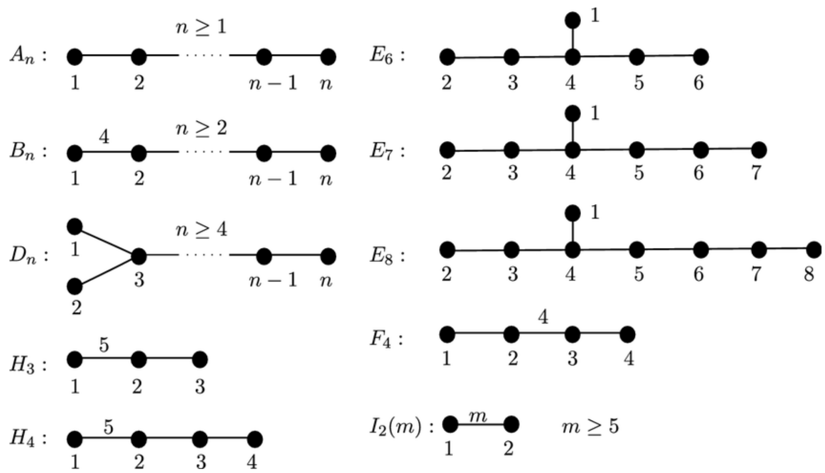
The *rank* of (W, S) (or just W, when S is understood) is $|S|$.

In the following, we'll restrict our focus to the world of finite Coxeter groups. We'll also use subscripts on our groups to denote the rank of the Coxeter system.

Theorem 2.7 (Coxeter, 1935)**.** *The following*[3] *is a classification of the finite irreducible Coxeter groups:*

(1) A_n *(a.k.a.,* S_{n+1}*)*
(2) B_n
(3) D_n
(4) E_6
(5) E_7
(6) E_8
(7) F_4
(8) G_2 *(a.k.a.,* $I_2(6)$*)*
(9) H_3
(10) H_4
(11) $I_2(m)$.

The following are their Coxeter graphs:



Often, when proving results about finite irreducible Coxeter groups there are two main types of arguments: one can prove things case by case, for each type, or one's proof can be *type uniform*—where one's arguments do not depend on the type of the Coxeter group.

2.3. **Reduced Words.** Let (W, S) be a Coxeter system. Every element $w \in W$ can be written as a product of simple generators. A natural question is the following: how can we write $w \in W$ as a product of simple reflections using the fewest simples possible?

Definition 2.8. A *reduced word* (also a *reduced decomposition* or a *reduced expression*) for w is a word over S that represents w and uses the minimum possible number of simple generators. The minimum number of simples is called the *length* of w and is denoted $\ell(w)$.

Lemma 2.9. *For* $w \in W$ *and* $s \in S$*, we have* $\ell(sw) = \ell(w) \pm 1$.

Proof. Define a map $W \longrightarrow \{1, -1\} \simeq \mathbb{Z}/2\mathbb{Z}$ by $w \mapsto (-1)^{\ell(w)}$. This map is a group homomorphism (prove this as an exercise, or read the book!), so

$$(-1)^{\ell(sw)} = (-1)^{\ell(w)}(-1)^{\ell(w)} = (-1)^{\ell(w)+1}.$$

[3]The reason for writing G_2 separately, even though $G_2 \simeq I_2(6)$, is that G_2 appears as the Weyl group of the root system of a semisimple Lie algebra. The connection to Lie algebras is also the reason that the notation C_n is not used.

It follows that $\ell(sw) \equiv \ell(w)+1 \bmod 2$, but $\ell(sw) \le \ell(w)+1$. Similarly, $\ell(w) = \ell(ssw) \le \ell(sw)+1$, forcing $\ell(sw) \ge \ell(w) - 1$. $\square$

Example 2.10. In S_3, we have

Element	Reduced Word	Length
123	$\emptyset$	0
132	s_2	1
213	s_1	1
312	s_2s_1	2
231	s_1s_2	2
321	$s_1s_2s_1 = s_2s_1s_2$	3

3. Tuesday September 12

3.1. More about reduced words; Matsumoto's Theorem.

A *word* over S is a finite sequence of elements of S. A word representing $w \in W$ is *reduced* if it has the minimum possible length. This *length* of w is the length of the reduced word and is denoted $\ell(w)$. Recall Example 2.10.

A *nil move* deletes some ss in a word. A *braid move* replaces $(ss')^{m(s,s')} = ss'ss'\cdots$ with $(s's)^{m(s,s')} = s'ss's\cdots$.

Theorem 3.1 (Matsumoto's Theorem, or Tits Lemma)**.** *Any word can be transformed into a reduced word using only nil moves and braid moves.*

We'll see how all of this works in the symmetric group.

Theorem 3.2. *For $w \in S_n$, we have $\ell(w) = \operatorname{inv}(w)$.*

Proof. We induct on $\operatorname{inv}(w)$. Let $s_{i_1}\cdots s_{i_k}$ be a reduced word for w, and let $u_j = s_{i_1}\cdots s_{i_j}$. Note that u_j is obtained from u_{j-1} by reversing two adjacent numbers. It follows that $\operatorname{inv}(u_j) \le \operatorname{inv}(u_{j-1})+1$. So $\operatorname{inv}(w) = \operatorname{inv}(u_k) \le k = \ell(w)$.

Now, for the reverse inequality: let r be a descent of w. Let $v = ws_r$, and note that $\operatorname{inv}(v) = \operatorname{inv}(w) - 1$. By the inductive hypothesis, $\ell(v) = \operatorname{inv}(v)$, implying

$$\ell(w) = \ell(vs_r) \le \ell(v) + 1 = \operatorname{inv}(v) + 1 = \operatorname{inv}(w)$$

as desired. $\square$

A *reflection* in W is an element that is conjugate to a simple reflection. Often, we will use T to denote the set of reflections in W, i.e., $T = \{wsw^{-1} \mid s \in S,\ w \in W\}$. In S_n, a reflection is just a transposition.[4] Let $T_L(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$; similarly, let $T_R(w) = \{t \in T \mid \ell(wt) < \ell(w)\}$. We call $T_L(w)$ and $T_R(w)$ left and right inversions of w, respectively.

Example 3.3. Consider S_n, and suppose $1 \le i < j \le n$. For $w \in S_n$, we have $(ij) \in T_L(w)$ if and only if $w^{-1}(i) > w^{-1}(j)$. Similarly, $(ij) \in T_R(w)$ if and only if $w(i) > w(j)$.

Remark 3.4. For any $w \in W$, we have $\ell(w) = \ell(w^{-1})$. Hence $T_L(w) = T_R(w^{-1})$. That is, $\ell(tw) < \ell(w)$ if and only if $\ell(w^{-1}t) < \ell(w^{-1})$.

Theorem 3.5 (Strong Exchange Property)**.** *Suppose $w = s_1\cdots s_k$ for some $s_1,\dots,s_k \in S$. For $t \in T_L(w)$, we have $tw = s_1\cdots\widehat{s_i}\cdots s_k$ for some i, where the hat-over-an-element notation indicates that this element should be deleted.*

[4] In S_n, the conjugacy class of $\sigma \in S_n$ is the set of those elements with the same cycle type.

Example 3.6. We illustrate the theorem using an example. Consider $2431 \in S_4$. Then $s_1s_3s_2s_3$ is a reduced word for 2431. Let $t = (14) \in T_L(2431)$, and note that $tw = 2134 = s_1 = s_1s_3\widehat{s_2}s_3 = s_1s_3s_3$.

Corollary 3.7. *Let $s_1 \cdots s_k$ be a reduced word for $w \in W$. Then*

$$T_L(w) = \{s_1, s_1s_2s_1, s_1s_2s_3s_2s_1, \cdots, s_1s_2 \cdots s_{k-1}s_ks_{k-1} \cdots s_2s_1\}.$$

Proof. If $t \in T_L(w)$, then $tw = s_1 \cdots \widehat{s_i} \cdots s_k$. Rewriting this gives us $ts_1 \cdots s_is_{i+1} \cdots s_k = s_1 \cdots s_{i-1}s_{i+1} \cdots s_k$. Hence, $ts_1 \cdots s_i = s_1 \cdots s_{i-1}$, forcing $t = s_1 \cdots s_{i-1}s_is_{i-1} \cdots s_1$.

For the converse, if $t = s_1 \cdots s_{i-1}s_is_{i-1} \cdots s_1$, then $tw = s_1 \cdots \widehat{s_i} \cdots s_k$, so $\ell(tw) < \ell(w)$, telling us that $t \in T_L(w)$. □

A consequence of the above is that $|T_L(w)| = |T_R(w)| = \ell(w)$.

Let $D_L(w) = T_L(w) \cap S$ and $D_R(w) = T_R(w) \cap S$. The set $D_L(w)$ are called the *left descents* of w (similarly $D_R(w)$ are the *right descents*).

Example 3.8. A right descent of $w \in S_n$ is a simple transposition s_i such that $\ell(ws_i) < \ell(w)$, i.e., $\mathrm{inv}(ws_i) < \mathrm{inv}(w)$. Thus, $D_R(w) = \{s_i \in S \mid w(i) > w(i+1)\}$. Similarly, $D_L(w) = \{s_i \in S \mid w^{-1}(i) > w^{-1}(i+1)\}$.

Theorem 3.9 (Deletion Property). *If $w = s_1 \cdots s_k$, where $k > \ell(w)$, then $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k$ for some i and j.*

Proof. Choose i maximal so that $s_i \cdots s_k$ is not reduced. Then $s_{i+1} \cdots s_k$ is reduced, so $\ell(s_{i+1} \cdots s_k) = k - i$, and $\ell(s_1 \cdots s_k) < k - i + 1$. It follows that $\ell(s_i \cdots s_k) < \ell(s_{i+1} \cdots s_k)$. Thus, $s_i \cdots s_k = s_{i+1} \cdots \widehat{s_j} \cdots s_k$ for some j. Then $w = s_1 \cdots s_{i-1}s_i \cdots s_k = s_1 \cdots s_{i-1}s_{i+1} \cdots \widehat{s_j} \cdots s_k$. □

3.2. **Posets.**

Definition 3.10. A *poset* (partially ordered set) is a pair $(P, \leq)$ such that P is a set and $\leq$ is a relation on P that is:

(1) reflexive: $x \leq x$ for all $x \in P$;
(2) antisymmetric: $x \leq y$ and $y \leq x$ implies $x = y$ for $x, y \in P$;
(3) transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in P$.[5]

We write $x < y$ if $x \leq y$ and $x \neq y$. When the partial order on P is understood, we will drop the cumbersome notation $(P, \leq)$ for our poset and simply refer to the poset as P. A *chain* is a totally ordered set (i.e., a chain is a poset where all elements are comparable).

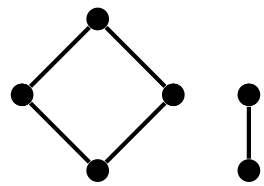
Let $x, y \in P$ with $x \leq y$. The *interval* between x and y is $[x, y] = \{z \in P \mid x \leq z \leq y\}$. We say y *covers* x if $\#[x, y] = 2$; in this case, we write $x \lessdot y$.

The *Hasse diagram* of P is a graphical representation of P. The diagram itself is the graph representing elements of P as vertices with edges given by cover relations (i.e., $x \lessdot y$ implies the existence of an edge between x and y). We orient the graph in the plane so that if $x \leq y$, then the vertex representing x is drawn below that of y.

Example 3.11. Consider a chain with 4 elements. Its Hasse diagram looks like

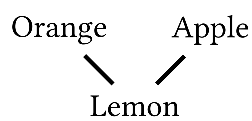
[5] My favorite intuitive way to explain posets to people who don't study math is by explaining the following example that everyone is already familiar with. Fruit forms a poset. For example, oranges are better than lemons (most people, maybe except Colin, don't eat lemons in the same way they eat oranges), but apples and oranges can't be compared!

Example 3.12. Here's another poset and its Hasse diagram:



4. Thursday September 14

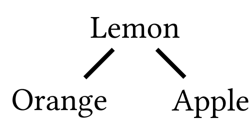
4.1. **Posets.** Eliot's Fruit Order: the underlying set is {apple, orange, lemon}, with order relations orange≥lemon, apple≥lemon, but apples and oranges can't be compared.



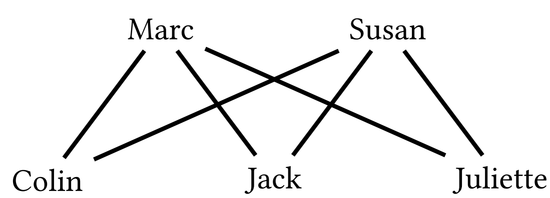
We'll use this example to define the *dual* of a poset.

Definition 4.1. The *dual* of a poset $(P, \leq)$ is the poset $(P, \leq')$ such that $x \leq y$ if and only if $y \leq' x$.

Colin's Fruit Order: the underlying set of fruit is the same, but the order relations are lemon≥orange and lemon≥apple, but again apples and oranges can't be compared. Note that Colin's Fruit order is dual to Eliot's.

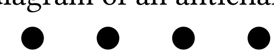


Here's another real-world example of a poset. Given a person A, we say that $A < B$ if A is a descendant of B.



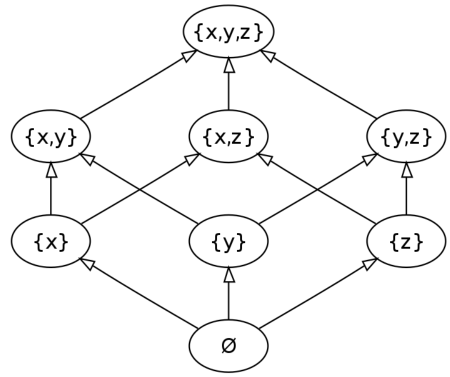
An *antichain* of size n is a poset of size n in which any two elements are incomparable.

Example 4.2. For $n = 4$, the Hasse diagram of an antichain looks like:



The nth *Boolean lattice* (we'll discuss lattices later) is the collection of subsets of $[n]$ ordered by containment. In other words, $I \leq J$ if and only if $I \subseteq J$.

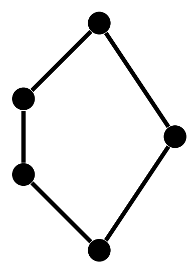
Example 4.3. The graphic below depicts the boolean lattice for $n = 3$:



We illustrate the definition of an interval in the poset in the figure below. Make figure.

A poset P is *graded* if there is a rank function $\mathrm{rk} : P \to \mathbb{Z}$ such that $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ whenever $x \lessdot y$.

Example 4.4. The following poset is not graded (try constructing a rank function), while the Boolean lattice is (do you see what the rank function on the Boolean lattice is?).



4.2. **The Bruhat Order.** The *Bruhat order* is a partial order that we will define on a Coxeter group. Let (W, S) denote a Coxeter system, and let $T = \{wsw^{-1} \mid s \in S, w \in W\}$ be the set of reflections. The *Bruhat order* on W is the partial order in which $u \le v$ whenever there exist reflections $t_1, \dots, t_k \in T$ such that $v = ut_1 \cdots t_k$ and $\ell(ut_1 \cdots t_{i-1}) < \ell(ut_1 \cdots t_i)$ for all $1 \le i \le k$.

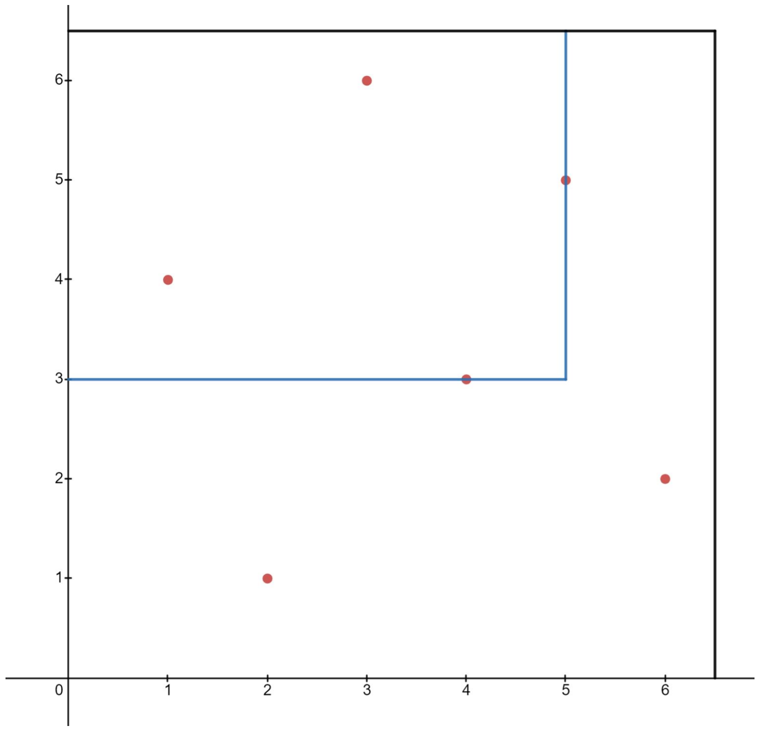
Example 4.5. Consider $I_2(5)$, the dihedral group of order 10. If $\{r, s\}$ is the set of simple reflections, recall the Coxeter graph of $I_2(5)$ is the graph with an edge between r and s with label 5. The following depicts the weak Bruhat order on $I_2(5)$. Add figure.

We have $rs = s(srs)$, so $rs \ge s$ and $rs \ge r$; we can similarly justify $sr \ge r$ and $sr \ge s$. We can use analogous arguments to justify the rest of the picture.

Remark 4.6. The Bruhat order on S_n corresponds to containment of Schubert varieties (i.e., the Bruhat order is not some random order on W we just pulled out of thin air—it actually comes from geometry).

Remark 4.7. There is a unique minimal element of W with respect to the Bruhat order—the identity e. (Justify this to yourself!)

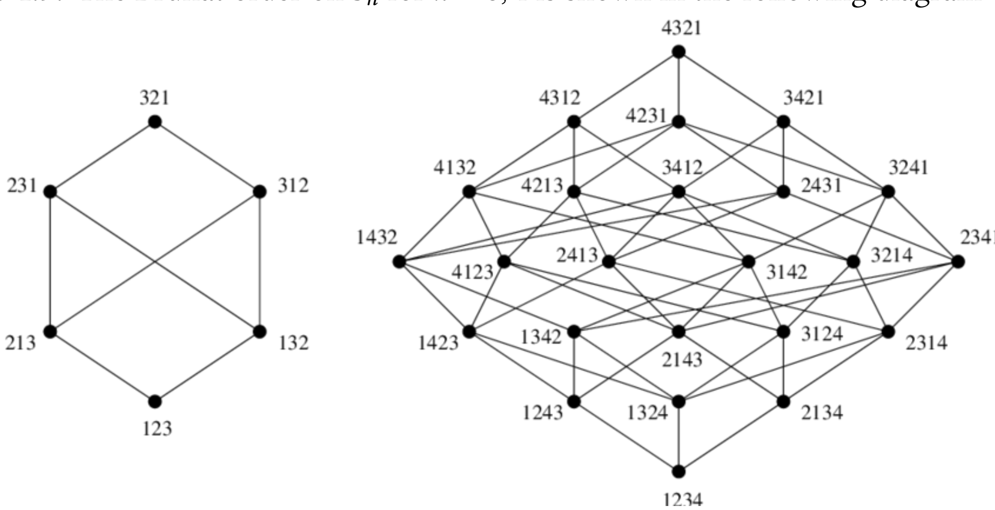
Next, we'll study the Bruhat order on S_n. For $x \in S_n$ and $i, j \in [n]$, let $x[i, j] = \#\{\alpha \in [i] \mid x(\alpha) \ge j\}$. We can visualize this in the following way. For a permutation $\sigma \in S_n$, we can plot $(i, \sigma(i))$ in $\mathbb{Z}^2$. Then $x[i, j]$ is the number of dots in the box enclosed by $y \ge j$ and $x \le i$. For example, if $x = 416352 \in S_6$, its plot is given by the red dots in the figure below. Moreover we see that $x[5, 3] = 4$, since there are 4 dots in the region enclosed by the blue lines.



Theorem 4.8. *For $x, y \in S_n$, we have $x \leq y$ if and only if $x[i,j] \leq y[i,j]$ for all $i, j \in [n]$.*

Proof. Suppose $y = xt$ for some $t \in T$ with $\ell(x) < \ell(y)$. Let $t = (a\ b)$ for $a < b$. Then we have $x(a) < x(b)$, and it is not so difficult ot see that $x[i,j] \leq y[i,j]$ for all $i, j \in [n]$. Thus, this is true whenever $x \leq y$. □

Example 4.9. The Bruhat order on S_n for $n = 3, 4$ is shown in the following diagram:



A *subword* of a word $s_1 \cdots s_k$ is a word $s_{i_1} \cdots s_{i_m}$, where $1 \leq i_1 < \cdots < i_m \leq k$.

Theorem 4.10 (Subword Property)**.** *For $u, v \in W$, the following are equivalent:*

(1) *$u \leq v$;*
(2) *every reduced word for v contains a reduced word for u as a subword;*
(3) *some reduced word for v contains a reduced word for u as a subword;*

We omit the proof and instead refer the reader to the textbook.

Corollary 4.11. *For $u, v \in W$, we have $u \le v$ if and only if there exist reflections $t_1, \ldots, t_k$ such that $v = t_k \cdots t_1 u$ and $\ell(t_i \cdots t_1 u) > \ell(t_{i-1} \cdots t_1 u)$ for all $1 \le i \le k$. Hence, the map $w \mapsto w^{-1}$ is an automorphism of the Bruhat order.*[6]

Corollary 4.12. *It follows that $\#[e, w] \le 2^{\ell(w)}$ under the Bruhat order. (This bound is tight—consider the Boolean lattice.)*

Theorem 4.13 (Chain Property)**.** *If $u < v$ in the Bruhat order, then there is a chain $u = x_0 < x_1 < \cdots < x_k = v$ with $\ell(x_i) = \ell(x_{i-1}) + 1$ for all $1 \le i \le k$. So, $u \lessdot v$ if and only if $\ell(v) = \ell(u) + 1$ and $v = ut$ for some $t \in T$. Therefore, the Bruhat order is in fact a graded poset with rank function given by length.*

5. Tuesday September 19

5.1. Properties of the Bruhat Order.

Remark 5.1. For $u, v \in W$, we have $u \le v$ (in the Bruhat order) if and only if some reduced word for v contains a word representing u.

Proof. The Deletion Property (Theorem 3.9) does the trick. □

Lemma 5.2. *Suppose $u, w \in W$ are distinct, and let $s_1, \ldots, s_q$ be a reduced word for w that contains a reduced word for u as a subword. Then there exists some $v \in W$ such that*

(1) *$u < v$;*
(2) *$\ell(v) = \ell(u) + 1$;*
(3) *some reduced word for v is a subword of $s_1 \cdots s_2$.*

Proof. Of all reduced words $s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_k}} \cdots s_q$ for u, choose one such that i_k is minimal. Let $t = s_q s_{q-1} \cdots s_{i_k} \cdots s_{q-1} s_q$. Then $ut = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_{k-1}}} \cdots s_{i_k} \cdots s_q$, so $\ell(ut) \le \ell(u) + 1$. We claim that this inequality must be strict. Assuming that this is true, then setting $v = ut$ proves the lemma.

To prove that the inequality is strict, assume for a contradiction that $\ell(u) > \ell(ut)$. By the Strong Exchange Property (Theorem 3.5), either

$$t = s_q s_{q-1} \cdots s_p \cdots s_{q-1} s_q$$

for some $p > i_k$ or

$$t = s_q \cdots \widehat{s_{i_k}} \cdots \widehat{s_{i_j}} \cdots s_r \cdots \widehat{s_{i_j}} \cdots \widehat{s_{i_k}} \cdots s_q$$

for some $r < i_k$ with $r \ne i_j$.

Case 1:

$$w = wt^2 = s_1 \cdots s_q (s_q s_{q-1} \cdots s_p \cdots s_{q-1} s_q)(s_q s_{q-1} \cdots s_p \cdots s_{q-1} s_q) = s_1 \cdots \widehat{s_{i_k}} \cdots \widehat{s_p} \cdots s_q,$$

but this is ridiculous, since $\ell(w) = q$.

Case 2:

$$\begin{aligned} u = ut^2 &= (s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_k}} \cdots s_q)(s_q \cdots \widehat{s_{i_k}} \cdots \widehat{s_{i_j}} \cdots s_r \cdots \widehat{s_{i_j}} \cdots \widehat{s_{i_k}} \cdots s_q)(s_1 \cdots s_{i_k} \cdots s_q) \\ &= s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_r} \cdots \widehat{s_{i_j}} \cdots s_{i_k} \cdots s_q, \end{aligned}$$

which contradicts the minimality of i_k. □

[6]By automorphism of a poset we mean the following. Given two posets $(P, \le_P)$ and $(Q, \le_Q)$, an *isomorphism* between P and Q is a bijective map $\phi : P \to Q$ such that $x \le_P y$ if and only if $\phi(x) \le_Q \phi(y)$. An *automorphism* of a poset P is simply a isomorphism from P to itself.

Theorem 5.3 (The Lifting Property)**.** *Suppose $u < w$ and $s \in D_L(w) \setminus D_L(u)$. Then $u \leq sw$ and $su \leq w$.* [7]

Proof. Let $s_1 \cdots s_q$ be a reduced word for sw. Then $ss_1 \cdots s_q$ is a reduced word for w. By the Subword Property (Theorem 4.10), there exists a subword $s_{i_1} \cdots s_{i_k}$ of $ss_1 \cdots s_q$ that is a reduced word for u. Since $s \notin D_L(u)$, we have that $s_{i_1} \neq s$. Moreover, $ss_{i_1} \cdots s_{i_k}$ is a reduced word for su and it is a subword of $ss_1 \cdots s_q$. □

Let P be a poset. A *lower bound* (resp., *upper bound*) for $x, y \in P$ is an element $z \in P$ such that $z \leq x$ and $z \leq y$ (resp., $z \geq x$ and $z \geq y$).

Proposition 5.4. *Any elements $u, w \in W$ have a lower bound and an upper bound in Bruhat order.*

Proof. The identity is a lower bound for both u and w. To find an upper bound, we induct on $\ell(u) + \ell(w)$. If the $\ell(u) + \ell(w) = 0$, then $u = w = e$ can be upper-bounded by e. Assume that $\ell(u)+\ell(w) > 0$, and without loss of generality, assume that $\ell(u) > 0$. Let $s \in D_L(u)$. By induction, there exist $x \in W$ such that $x \geq su$ and $x \geq w$. If $s \in D_L(x)$, then $u \leq x$ by the Lifting Property, so x is an upper bound for u and w. If $s \notin D_L(x)$, then $u \leq sx$ by the Lifting Property, so sx is an upper bound for u and w (we have $w \leq x \leq sx$). □

When W is finite, we have the following proposition:

Proposition 5.5. *If W is finite, then there exists $w_0 \in W$ such that $w \leq w_0$ for all $w \in W$. Also, $D_L(w_0) = S$.*

Moreover, if there exists $x \in W$ such that $D_L(x) = S$, then W is finite and $x = w_0$.

Proof. For the first claim, note that the Bruhat order cannot have multiple maximal elements if W is finite, since they would have to have an upper bound. If $D_L(w_0) \neq S$, then there exists $s \in S$ such that $sw_0 \geq w_0$ contradicting the maximality of w_0.

For the second statement, suppose that $D_L(x) = S$. We will show that $u \leq x$ for all $u \in W$ by induction on $\ell(u)$. If $\ell(u) = 0$, then $u = e$, so $u \leq x$. Assume $\ell(u) > 0$. Let $s \in D_L(u)$. Then $su < u$ so $su \leq x$. But since $s \in D_L(x)$, the Lifting Property says that $u \leq x$. Then $W = [e, x]$ is finite (recall that $\#[e, x] \leq 2^{\ell(x)}$). □

Often, w_0 is called the *long element.*

5.2. **Properties of the Long Element.** Here are some facts about the long element:

(1) $w_0^2 = e$;
(2) $\ell(ww_0) = \ell(w_0w) = \ell(w_0) - \ell(w)$;
(3) $\ell(w_0ww_0) = \ell(w)$;
(4) $T_L(ww_0) = T \setminus T_L(w)$;
(5) $\ell(w_0) = |T|$.

Actually, the maps $x \mapsto xw_0$ and $x \mapsto w_0x$ are anti-automorphisms of the Bruhat order (i.e., $x \leq y$ if and only if $xw_0 \geq yw_0$ if and only if $w_0x \geq w_0y$), so the map $x \mapsto w_0xw_0$ is an automorphism of the Bruhat order.

In S_n, we have that $w_0 = n(n-1)\cdots 321$. Note that xw_0 is the reverse of x, and w_0x is the complement of x. Lastly, w_0xw_0 is the reverse complement of x.

Example 5.6. Consider $x = 416352 \in S_6$. Note that

(1) $xw_0 = 253614$;

[7] If you go to the gym, this should be your favorite theorem.

(2) $w_0x = 361425$;
(3) $w_0xw_0 = 524163$.

The elements covering e in the Bruhat order are simple reflections, so the map taking $s \mapsto w_0sw_0$ for $s \in S$ is an automorphism of the Coxeter graph.

6. Thursday September 21

6.1. **More on the Long Element.** Recall that the elements covering e in the Bruhat order are simple reflections, so the map taking $s \mapsto w_0sw_0$ for $s \in S$ is an automorphism of the Coxeter graph. Hence, the map $s \mapsto w_0sw_0$ for $s \in S$ is an automorphism of the Coxeter graph (i.e., a bijective map preserving adjacencies and labels on the edges).

Example 6.1. Consider S_n. The Coxeter graph is the path graph on $n-1$ vertices, where s_i and s_{i+1} are adjacent for $1 \le i < n-1$. As an exercise, check that $w_0s_iw_0 = s_{n-i}$.

Example 6.2. Consider B_n (for $n \ge 3$). Here we see that $w_0s_iw_0 = s_i$, i.e., the automorphism induced by the Coxeter graph is the identity.

Example 6.3. Consider D_n for $n \ge 3$. Here, we have $w_0s_iw_0 = s_i$ for $2 \le i \le n-1$. We also have

$$w_0s_0w_0 = \begin{cases} s_0 & \text{if } n \text{ is even;} \\ s_1 & \text{if } n \text{ is odd.} \end{cases} \quad \text{and} \quad w_0s_1w_0 = \begin{cases} s_1 & \text{if } n \text{ is even;} \\ s_0 & \text{if } n \text{ is odd.} \end{cases}$$

Every automorphism of the Coxeter graph yields an automorphism of the Coxeter group, and an automorphism of Bruhat order on W.

Theorem 6.4. *Suppose (W,S) is irreducible and $|S| \ge 3$. If $\varphi : W \to W$ is an automorphism of Bruhat order fixing all of the simple reflections (i.e., $\varphi(s) = s$ for all $s \in S$), then either $\varphi(x) = x$ for all $x \in W$ or $\varphi(x) = x^{-1}$ for all $x \in W$.*

6.2. **Parabolic Subgroups and Quotients.** Let $J \subset S$ be a subset of simple reflections, and let W_J be the subgroup of W generated by J. This is called a (standard)[8] *parabolic subgroup* of W.

Proposition 6.5. *Let W_J be a parabolic subgroup of W. Then*
(1) (W_J, J) is a Coxeter system;
(2) for all $w \in W_J$, we have $\ell_J(w) = \ell(w)$;
(3) $W_I \cap W_J = W_{I\cap J}$;
(4) $\langle W_I \cup W_J \rangle = W_{I\cup J}$;
(5) $W_I = W_J$ if and only if $I = J$.

If W_J is finite, it has a long element, which we will denote by $w_0(J)$.
Now, let $D_I^J = \{w \in W \mid I \subset D_R(w) \subset J\}$; let $W^J = D_\emptyset^{S\setminus J}$; and let $D_I = D_I^I$.

Proposition 6.6. *Every $w \in W$ has a unique factorization $w = w^J w_J$ with $w^J \in W^J$ and $w_J \in W_J$. Moreover, $\ell(w) = \ell(w^J) + \ell(w_J)$.*

Example 6.7. Let $W = S_9$ and $J = \{s_1, s_2, s_4, s_7\}$. Let $w = 426915783$. We have $W^J = 246195783$ and $w_J = 213546789$.

[8]Some define parabolic subgroups to be the subgroups conjugate to standard parabolic subgroups (i.e., parabolic subgroups as we have defined them above).

Corollary 6.8. *Every coset wW_J has a unique representative of minimum length (and it is w^J). These minimum-length coset representatives are the elements of W^J.*

Proof. If $uW_J = wW_J$, then $w = uv$ for some $v \in W_J$. Write $u = u^J u_J$ and $u = w^J w_J$. Then $w = w^J w_J = u^J u_J v$. So $w^J = u^J$ by the uniqueness of the factorization from Proposition 6.6. □

Corollary 6.9. *If W_J is finite, then wW_J has a unique element of maximum length; it is $w^J w_0(J)$. These maximum-length coset representatives are the elements of D_J^S.*

Example 6.10. Again let $W = S_9$ with $J = \{s_1, s_2, s_4, s_7\}$ and $w = 426915783$. We have $w_0(J) = \underline{321}\,\underline{54}6\underline{87}9$ and $w^J w_0(J) = 642915873$. Note that we can get $w^J w_0(J)$ by "putting the things you can" into descending order.

There are also mirrored versions of the objects defined above:

$$^{J}W = \{w \in W \mid D_L(w) \subset S \setminus J\}$$

is the set of minimum-length coset representatives of right cosets of W_J. Every $w \in W$ can be factored uniquely as $w = w_J\,{}^J w$ with $w_J \in W_J$ and ${}^J w \in {}^J W$ (not the same w_J as before). (We get this from $\{w^{-1} \mid w \in W_J\} = W_J$ and $\{w^{-1} \mid w \in W^J\} = {}^J W$.)

We can restrict Bruhat order to W^J to get an interesting poset, called a *parabolic quotient.* Define $P^J : W \to W^J$ by $P^J(w) = w^J$, i.e., the minimum-length representative of wW_J.

Proposition 6.11. *We have that P^J is order-preserving. In other words, $u \le v$ if and only if $P^J(u) \le P^J(v)$.*

Corollary 6.12. *Any two elements of W^J have an upper bound in the Bruhat order.*

Proof. Let $u, v \in W^J$. Find $w \in W$ with $u \le w$ and $v \le w$. Then $u = P^J(u) \le P^J(w)$ and $v = P^J(v) \le P^J(w)$. □

If W^J is finite, then it has a maximal element w_0^J.

Proposition 6.13. *Suppose W is finite and $J \subset S$. The map $\alpha : W^J \to W^J$ defined by $\alpha(x) = w_0 x w_0(J)$ is an antiautomorphism of Bruhat order of W^J.*

Theorem 6.14. *If $u < w$ in W^J, then there exist $x_0, x_1, \ldots, x_k \in W^J$ with $u = x_0 < x_1 < \cdots < x_k = w$ and $\ell(x_i) = \ell(x_{i-1}) + 1$ for all i.*

Corollary 6.15. *The Bruhat order on W^J is graded.*

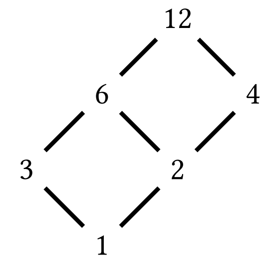
6.3. **Lattices.** Let P be a poset. For $x, y, z \in P$, we say that z is the *greatest lower bound*, or *meet*, of x and y if z is the unique maximal element of $\{u \in P \mid u \le x, y\}$. Note that such an element z need not exist; when it does, we denote it by $x \wedge y$. The *least upper bound*, or *join*, is defined dually and is denoted $x \vee y$.

Definition 6.16. A *lattice* is a poset L such that any two elements $x, y \in L$ have a meet and a join.

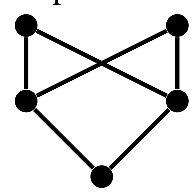
When working with lattices we can regard them as order-theoretic objects (i.e., as posets with some additional properties) or as algebraic objects (i.e., we can think of $\wedge$ and $\vee$ as algebraic operations).

Example 6.17. The *Boolean lattice* of subsets of $[n]$ ordered by containment is a lattice (recall this from Example 4.3). Here, $x \wedge y = x \cap y$; similarly, $x \vee y = x \cup y$.[9]

Example 6.18. The divisors of some positive integer N under divisibility is a lattice. Here, $x \wedge y = \gcd(x, y)$, and $x \vee y = \mathrm{lcm}(x, y)$. Note that the Boolean lattice on $2^{[n]}$ is the lattice of divisors of the product of n primes. For example, for $N = 12$ we have

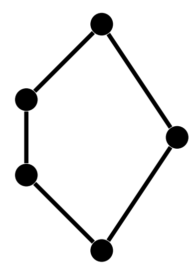


Example 6.19. The following is a nonexample of a lattice:

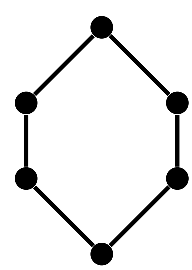


A lattice L is called *distributive* if $x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z, \in L$.

Example 6.20. The following lattice is not distributive:



The following is also not distributive:

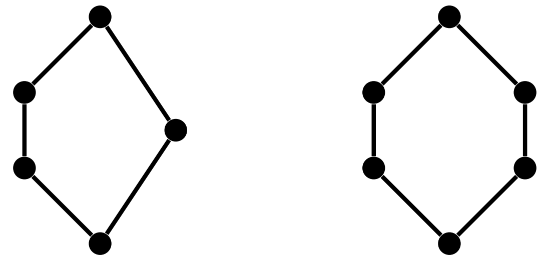


7. Tuesday September 26

7.1. **More Lattices.** Recall the definition of a distributive lattice: A lattice L is called *distributive* if $x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z, \in L$.

Example 7.1. The following lattices are not distributive. As an exercise, justify to yourself why this is the case.

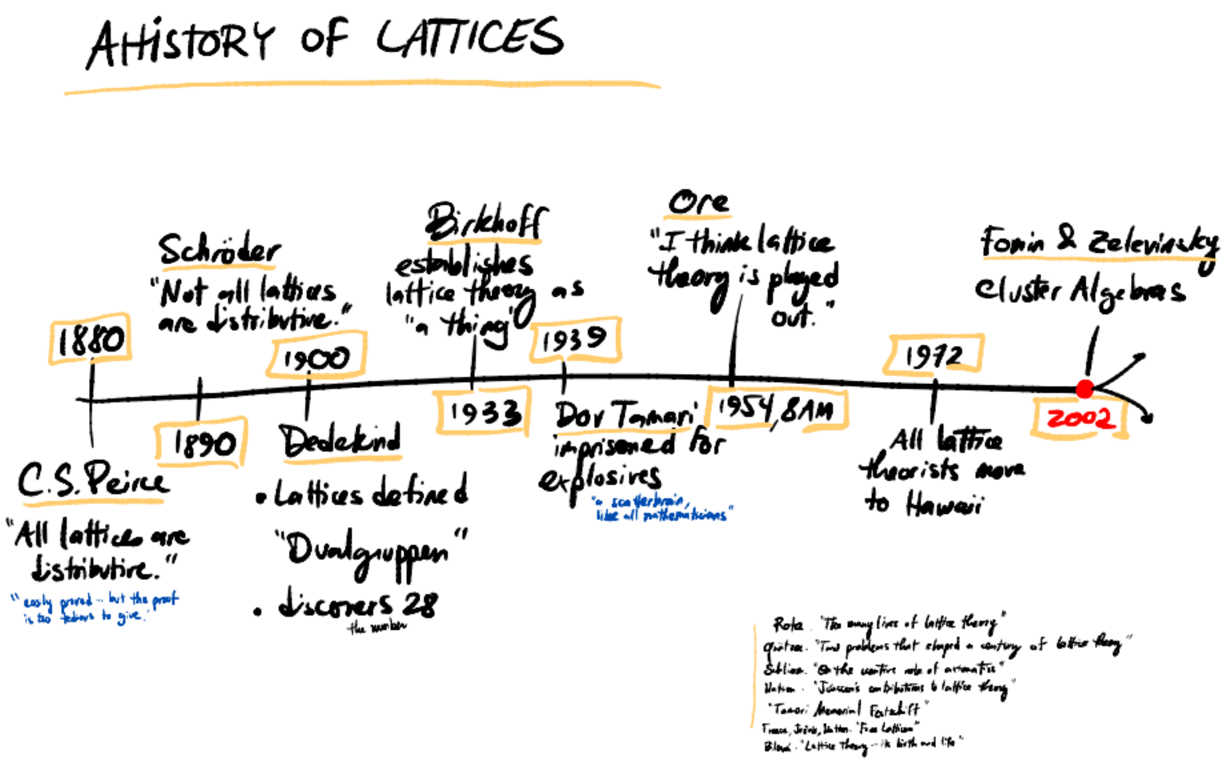
[9] Remembering this example is a good mnemonic for remembering that $\wedge$ denotes the greatest lower bound and that $\vee$ denotes the least upper bound.



Distributive lattices are not just natural from an algebraic viewpoint—distributivity is a natural property from an order-theoretic vantage point. Let P be a poset. An *order ideal* of P is a subset $I \subset P$ such that for all $x, y \in P$, we have $y \in I$ and $x \leq y$ imply $x \in I$. We say some subset $S \subset P$ generates an order ideal I if I is the smallest order ideal containing S. An order ideal generated by a single element is said to be *principal.* Let $J(P)$ be the set of order ideals of P, ordered by containment. Then $J(P)$ is a poset, and the subposet given by the principal order ideals is isomorphic to P.

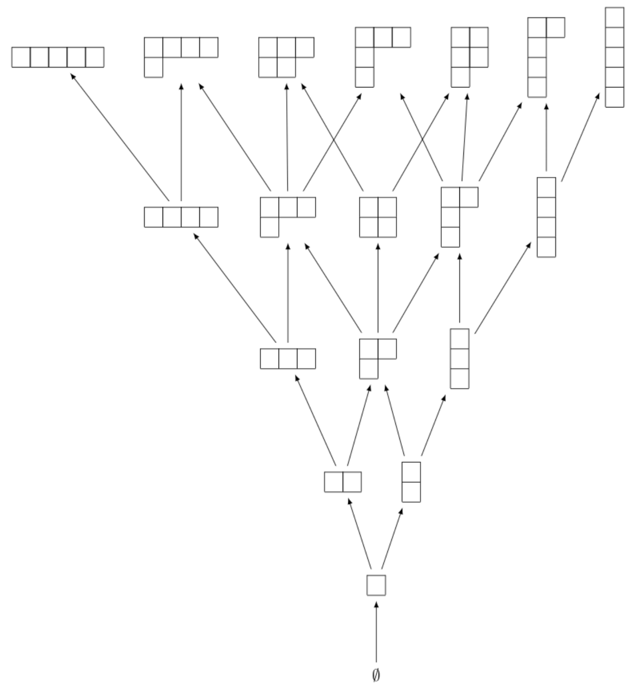
Example 7.2. Consider the poset P on three elements that has a unique minimal element and is not a chain. What is $J(P)$? Add picture later.

Theorem 7.3 (Fundamental Theorem of Finite Distributive Lattices; Birkhoff, 1937)**.** *If P is a finite poset, then $J(P)$ is a distributive lattice. If L is a finite distributive lattice, then there exists a (unique) finite poset P such that $L \simeq J(P)$.*



This is a summary of the history of lattice theory due to Nathan Williams.

Example 7.4. Young's Lattice is the graded lattice whose elements are partitions (viewed as Young diagrams) and whose order relations is given by containment (of Young diagrams). Young's Lattice is a distributive, infinite lattice.



Let L be a finite lattice. Every subset $X \subset L$ has a meet, denoted $\bigwedge X$, and a join, denoted $\bigvee X$. So L has a unique minimal element, which we denote by $\widehat{0} = \bigwedge L$. Likewise, L has a unique maximal element, denoted $\widehat{1} = \bigvee L$.

Remark 7.5. Note that $\bigwedge\{x\} = x$ and $\bigvee\{x\} = x$. Moreover, $\bigwedge \emptyset = \widehat{1}$ and $\bigvee \emptyset = \widehat{0}$.

An element j of a lattice L is *join-irreducible* if j is not the join of a finite subset $X \subset L$ with $j \notin X$. A *meet-irreducible* element is defined dually.

In the lattice of divisors of N, the join-irreducible elements are the powers of primes dividing N. In Young's Lattice, the join-irreducible elements are the rectangular-shaped Young diagrams.

Proposition 7.6. *Every element of a finite lattice can be written as a join of some set of join-irreducible elements.*

Proof. Let L be a finite lattice, and let $x \in L$. If x is join-irreducible, then we are done. Otherwise $x = \bigvee X$ for some set $X \subset L$ with $x \notin X$. By induction, each element of X is a join of some join-irreducibles. Hence, so is x. □

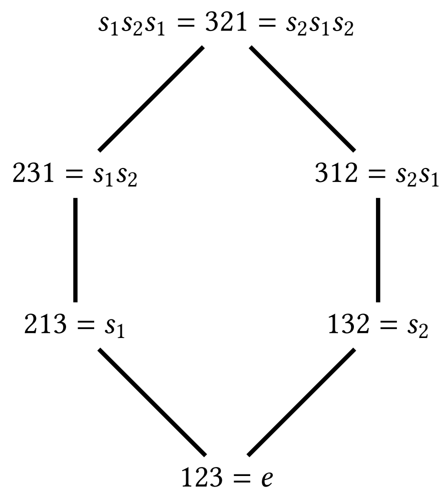
Proposition 7.7. *If p^α is a prime power that divides* $\operatorname{lcm}(m_1, \ldots, m_k)$*, then $p^\alpha | m_i$ for some i.*

7.2. **Weak Order and Reduced Words.** Let (W, S) be a Coxeter system. For $u, v \in W$, write $u \leq_R v$ if there exists $x \in W$ such that $v = ux$ and $\ell(v) = \ell(u) + \ell(x)$. Equivalently, $u \leq_R v$ if and only if there exists a reduced word for v that contains a reduced word for u as a prefix. The order $\leq_R$ is called the *right weak order.*

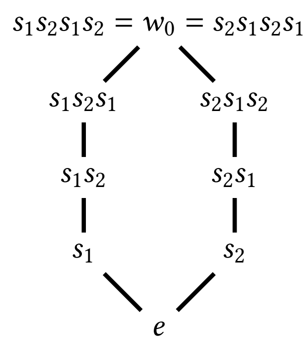
Write $u \leq_L v$ if there exists $x \in W$ such that $v = xu$ and $\ell(v) = \ell(x) + \ell(u)$. Equivalently, $u \leq_L v$ if and only if there exists a reduced word for v that contains a reduced word for u as a suffix. The order $\leq_L$ is called the *left weak order.*

Remark 7.8. The posets $(W, \leq_R)$ and $(W, \leq_L)$ are isomorphic via the map $w \mapsto w^{-1}$.

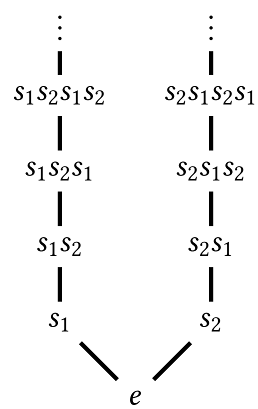
Example 7.9. Consider $W = S_3$:



Example 7.10. Consider the weak order on $I_2(4)$ with $S = \{s_1, s_2\}$.



Example 7.11. Consider the weak order on $I_2(\infty)$ with $S = \{s_1, s_2\}$.



Proposition 7.12. *Let (W, S) be a Coxeter system. For $u, v \in W$, we have $u \leq_R v$ if and only if $T_L(u) \subset T_L(v)$.*

Proof idea. If $s_1 \cdots s_q$ is a reduced word for u, then $T_L(u) = \{s_1, s_1s_2s_1, s_1s_2s_3s_2s_1, \dots\}$. $\square$

Proposition 7.13. *Let W be finite. Then the maps $x \mapsto xw_0$ and $x \mapsto w_0x$ are antiautomorphisms of the right weak order. In other words, $x \leq_R y$ if and only if $yw_0 \leq_R xw_0$ if and only if $w_0y \leq_R w_0x$. The map $x \mapsto w_0xw_0$ is an automorphism of the weak order.*

Proof. Suppose $s \lessdot_R ws$. Then

$$\ell(w_0wsw_0) = \ell(w_0) - \ell(wsw_0) = \ell(w_0) - (\ell(w_0) - \ell(ws)) = \ell(ws) = \ell(w) + 1 = \ell(w_0ww_0) + 1,$$

which proves the last statement of the result.

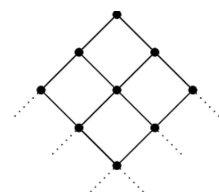
We'll finish the rest of the proof next time. □

8. Thursday September 28

8.1. More on the Weak Order.

Remark 8.1. View $\mathbb{N}^2$ as a poset, where $(a, b) \leq (a', b')$ if and only if $a \leq a'$ and $b \leq b'$. Young's lattice is isomorphic to the lattice of finite order ideals of $\mathbb{N}^2$.

Remark 8.2. Let L be the lattice below. This lattice has no join-irreducible elements.



Proposition 8.3. *For $u, v \in W$, then $u \leq_R v$ if and only if $T_L(u) \subset T_L(v)$.*

Proof. Suppose $T_L(u) \subset T_L(v)$. Let $s_1 \cdots s_k$ be a reduced word for u. Let $y_i = s_1 \cdots s_i$. We will prove by induction on i that $y_i \leq_R v$. If $i = 0$, then $y_i = e \leq_R v$.

Now, assume $1 \leq i \leq k$. Let $t_j = s_1 \cdots s_{j-1}s_js_{j-1} \cdots s_1$. Then $t_1, \ldots, t_k$ are the distinct elements of $T_L(u)$. By induction, there is some reduced word for v of the form $s_1 \cdots s_{i-1}s'_1 \cdots s'_q$, where $w = \ell(v) - i + 1$. Since $t_i \in T_L(u) \subset T_L(v)$ and $t_i \in \{t_1, \ldots, t_{i-1}\}$, we have $t_i = s_1 \cdots s_{i-1}s'_1 \cdots s'_m \cdots s'_1s_{i-1} \cdots s_1$ for some $1 \leq m \leq q$. Hence,

$$\begin{aligned} v = t_i^2 v &= (s_1 \cdots s_i \cdots s_1)(s_1 \cdots s_{i-1}s'_1 \cdots s'_m \cdots s'_1s_{i-1} \cdots s_1)v \\ &= (s_1 \cdots s_i \cdots s_1)(s_1 \cdots s_{i-1}s'_1 \cdots s'_m \cdots s'_q) \\ &= s_1 \cdots s_is'_1 \cdots \widehat{s_m}' \cdots s'_q. \end{aligned}$$

This word has length $i + q - 1 = \ell(v)$, so it is reduced. Hence, $y_i \leq_R v$. □

Proposition 8.4. *Let W be finite. Then the maps $x \mapsto xw_0$ and $x \mapsto w_0x$ are antiautomorphisms of the right weak order. In other words, $x \leq_R y$ if and only if $yw_0 \leq_R xw_0$ if and only if $w_0y \leq_R w_0x$. Then map $x \mapsto w_0xw_0$ is an automorphism of the weak order.*

Proof. It suffices to show that $x \mapsto xw_0$ and $x \mapsto w_0x$ are antiautomorphisms (we get the last statement for free). Suppose $u \leq_R v$. Equivalently, $\ell(u^{-1}v) = \ell(v) - \ell(u)$. So

$$\begin{aligned} \ell((w_0v)^{-1}(w_0u)) = \ell(v^{-1}u) = \ell((u^{-1}v)^{-1}) = \ell(u^{-1}v) &= \ell(v) - \ell(u) \\ &= (\ell(w_0) - \ell(u)) - (\ell(w_0) - \ell(v)) = \ell(w_0u) - \ell(w_0v). \end{aligned}$$

Thus, $w_0v \leq_R w_0u$. □

Proposition 8.5. *For $w \in W$ and $s \in S$, the following are equivalent:*

(1) *$s \in D_L(w)$;*
(2) *$s \leq_R w$;*
(3) *there exists a reduced word for w that begins with s.*

Proof. Note that (2) and (3) are equivalent by the definition of the right weak order.

(3) $\Longrightarrow$ (1): If $ss_1\cdots s_k$ is a reduced word for w, then $s_1\cdots s_k = sw$, so $\ell(sw) \le k < k+1 = \ell(w)$.

(1) $\Longrightarrow$ (3): Suppose $s \in D_L(w)$. Let $s'_1\cdots s'_r$ be a reduced word for sw. Then $ss'_1\cdots s'_r$ is a reduced word for w that begins with s. □

Proposition 8.6. *If $u \le_R v$, then $[u,v]_R \simeq [e,u^{-1}v]_R$. We will show that the map $f : [e,u^{-1}v]_R \to [u,v]_R$ given by $f(x) = ux$ is a poset isomorphism.*

Proof. We will show that the map $f : [e,u^{-1}v]_R \to [u,v]_R$ given by $f(x) = ux$ is a poset isomorphism.

We have

$$\ell(v) = \ell(u) + \ell(u^{-1}v) \le \ell(u) + \ell(x) + \ell(x^{-1}u^{-1}v) \tag{1}$$

and

$$\ell(v) \le \ell(ux) + \ell(x^{-1}u^{-1}v) \le \ell(u) + \ell(x) + \ell(x^{-1}u^{-1}v). \tag{2}$$

Now, $x \le_R u^{-1}v$ if and only if equality holds in (1), which is true if and only if equality holds in (2). This holds if and only if $u \le_R ux \le v$. Hence, $x \in [e,u^{-1}v]_R$ if and only if $f(x) \in [u,v]_R$. Note that f is a bijection.

For $x,y \in [e,u^{-1}v]_R$, a similar argument shows that $x \in [e,y]_R$ if and only if $ux \in [u,uy]_R$. So $x \le_R y$ if and only if $ux \le_R uy$.

□

A *meet-semilattice* is a poset L such that every finite subset of L has a meet. A meet-semilattice is called *complete* if every (not necessarily finite) subset of L has a meet. Note that Bruhat order is not a meet-semilattice—consider the Bruhat order on S_3.

Theorem 8.7. *The weak order $(W, \le_R)$ is a complete meet-semilattice.*

Proof. Suppose $x,y \in W$; note that it suffices to show $x \wedge y$ exists. We induct on $\ell(x)$. Let $E = [e,x]_R \cap [e,y] - R$. If $E = \{e\}$. Induct on $\ell(x)$. Let $E = [e,x]_R \cap [e,y]_R$. If $E = \{e\}$, then $e = x \wedge y$. We may assume $E \ne \{e\}$. Pick $z \in E$ of maximum length. We will show that $z = x \wedge y$; to do so, we need to prove that $w \le_R z$ for all $w \in E$.

Suppose $s \in E \cap S$. Let $z = s_1\cdots s_r$, $x = s_1\cdots s_r s'_1\cdots s'_p$, and $y = s_1\cdots s_r s''_1\cdots s''_q$ be reduced words. If $s \not\le_R z$, then by the Exchange Property, we have $x = ss_1\cdots s_r s'_1\cdots \widehat{s'_i}\cdots s'_p$ and $y = s_1\cdots s_r s''_1\cdots \widehat{s''_j}\cdots s''_q$ are reduced. So $ss_1\cdots s_r \in E$, but $\ell(ss_1\cdots s_r) = r+1 > r = \ell(z)$, which is a contradiction. So, $s \le_R z$.

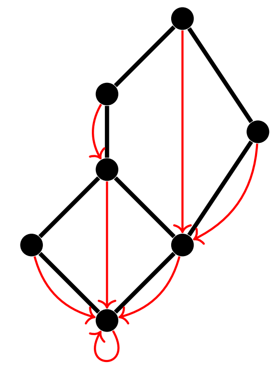
Now, let $w \in E \setminus \{e\}$. We use the following fact: if $x \in D_L(u) \cap D_L(v)$, then $u \le_R v$ if and only if $su \le_R sv$. Let $s \in D_L(w)$. Then $s \le_R w \le_R x$ and $s \le_R w \le_R y$. So $s \in D_L(x) \cap D_L(y)$. Also, $s \in D_L(z)$ by our work in the above. Since $\ell(sx) < \ell(x)$, we apply the inductive hypothesis to sx, which tells us that $z' := sx \wedge sy$ exists. Applying the aforementioned fact to $u = x$ and $v = w$, we have $sw \le_R sx$; similarly, $sw \le_R sy$. It follows that $sw \le_R z'$. To be continued... □

9. Tuesday October 3

10. Thursday October 5

10.1. **Pop.** Let L be a locally finite meet-semilattice. Define $\mathrm{Pop} : L \to L$ by $\mathrm{Pop}(x) = \wedge(\{x\} \cup \{y \in L \mid r \lessdot x\})$.

Example 10.1. Consider Pop on the following example

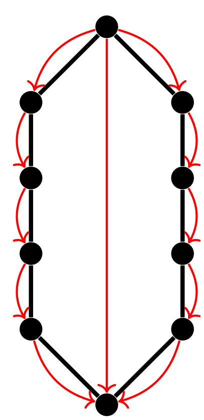


Suppose W is finite. The *Coxeter number* of W is $h = 2|T|/|S|$. When $W = S_n$, $h = 2\binom{n}{2}/(n-1) = n$. Since we can classify the finite irreducible Coxeter groups, we can write down all of their Coxeter numbers:

W	$A_{n-1} = S_n$	B_n	D_n	E_6	E_7	E_8	F_4	H_3	H_4	$I_2(m)$
h	n	$2n$	$2n-2$	12	18	30	12	10	30	m

Theorem 10.2 (Defant, 2022)**.** *The maximum number of iterations of* Pop *needed to send an element of W to e is $h-1$. If $J = S \setminus \{s\}$ for some $s \in S$, then $w_0(J)w_0$ requires $h-1$ iterations of* Pop *to reach e.*

Example 10.3. Consider $I_2(5)$.



10.2. S_n **Geometrically.** For $1 \leq i \leq j \leq n$, let $H_{ij} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$. The collection of hyperplanes H_{ij} is called the (nth) *braid arrangement.* Reflecting through H_{ij} has the effect of swapping the ith and jth coordinates. So S_n acts on $\mathbb{R}^n$ by permuting coordinates: $w(x_1, \dots, x_n) = (x_{w^{-1}(1)}, \dots, x_{w^{-1}(n)})$. So S_n can be identified with the group generated by reflections through hyperplanes in the braid arrangement.

A *set partition* of $[n]$ is a collection of disjoint subsets of $[n]$ whose union is $[n]$. The subsets making up a set parition are called *blocks.* The number of set partitions of $[n]$ is called the nth *Bell number.* It can be shown that

$$\sum_{n \geq 0} \mathrm{Bell}_n \frac{x^n}{n!} = e^{e^x - 1}.$$

Moreover, $\mathrm{Bell_n}$ is the number of rhyme schemes of a poem with n lines. For example, a poem with the rhyme scheme AABA corresponds to $\{\{1,2,4\},\{3\}\}$. Set partitions of $[n]$ correspond to intersections of hyperplanes in the braid arrangement.

Example 10.4. Let $n = 6$. The following is a set partition of $[6]$: $\{\{1, 2, 5\}, \{3, 6\}, \{4\}\}$.[10]

Consider $H_{12} \cap H_{15} \cap H_{25} \cap H_{36}$ (note that H_{25} is redundant here). Note that

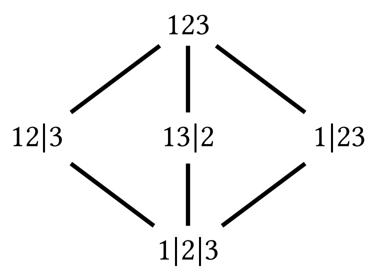
$$H_{12} \cap H_{15} \cap H_{25} \cap H_{36} = \{(x, x, y, z, x, y) \mid x, y, z \in \mathbb{R}\}.$$

Intersections of hyperplanes in the "Coxeter arrangement" of W are analogs of set partitions.

Fact 10.5. Intersections of hyperplanes in the Coxeter arrangement also correspond to conjugates of a standard parabolic subgroup. Suppose W acts on $\mathbb{R}^n$. If $U = wW_Jw^{-1}$, then the points in $\mathbb{R}^n$ fixed by all elements of U form an intersection of hyperplanes in the Coxeter arrangement.

Let $\prod_n$ be the set of set partitions of $[n]$. We endow $\prod_n$ with a partial order such that $\rho \leq \rho'$ if every block of ρ is contained in a block of ρ'. This is called the nth *partition lattice.* Here the meet is the "greatest common refinement" and the join is given by "combining blocks as minimally as possible."

Example 10.6. Consider $n = 3$.

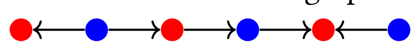


A *standard Coxeter element* of W is an element $c \in W$ obtained by multiplying all of the simple reflections in some order.

Example 10.7. In S_n, we have that $c = s_1 s_2 \cdots s_{n-1} = (12 \cdots n)$ is a standard Coxeter element. Another is

$$c' = \prod_{i \notin 2\mathbb{Z}} s_i \prod_{i \in 2\mathbb{Z}} s_i.$$

There correspond to *acyclic orientations* of the Coxeter graph. In S_7, we have



Exercise 10.8. Suppose the Coxeter graph of W is a tree. Show that all standard Coxeter elements are conjugate to each other.

If W is finite and irreducible, then all Coxeter elements are conjugate to each other. Hence, they all have the same order.

Fact 10.9. This order is the Coxeter number h. A Coxeter element is an element that is conjugate to a standard Coxeter element.

Let $T = \{wsw^{-1} \mid w \in W, s \in S\}$. This is called the *reflection length* of w. Let $\ell_T(w)$ be the minimum number of reflections needed to write w.

Example 10.10. For $w \in S_n$, we have $\ell_T(w) = n - \#\{\text{cycles in } w\}$. Thus, $\ell_T(e) = n - n = 0$ and $\ell_T((ij)) = n - (n - 1) = 1$.

Note that if u and v are conjugate, then $\ell_T(u) = \ell_T(v)$. The *absolute order* on W is defined so that $u \leq_{ab} v$ if and only if $\ell_T(u^{-1}v) = \ell_T(v) - \ell_T(u)$.

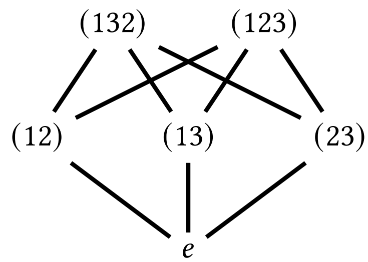
[10]Another variant of stack sorting exists for set partitions; this is called foot-sorting for socks.

11. Tuesday October 10

11.1. **Reflection.** Recall that a *reflection word* is a word over $T = \{wsw^{-1} \mid w \in W, s \in S\}$. The *reflection length* of $w \in W$, denoted $\ell_T(w)$ is the minimum length of a reflection word representing w. The *absolute order* on W is the partial order $\leq_{ab}$ defined so that $u \leq_{ab} v$ if and only if $\ell_T(u^{-1}v) = \ell_T(v) - \ell_T(u)$. If u and v are conjugate, then they have the same reflection length. Indeed, suppose $v = wuw^{-1}$ for some element w. If $u = t_1 \cdots t_r$ for $t_1 \cdots t_r \in T$, then $v = (wt_1w^{-1}) \cdots (wt_rw^{-1})$. Since $u^{-1}v$ and $v^{-1}u$ are conjugate ($u^{-1}v = v^{-1}(vu^{-1})v$), we have $u \leq_{ab} v$ if and only if $\ell_T(vu^{-1}) = \ell_T(v) - \ell_T(u)$.

Fact 11.1. Suppose W is finite. The maximal elements of the absolute order are the Coxeter elements.

Example 11.2. In S_3, we have the following:



where we note that

$$\ell_T((13)(123)) = \ell_T((13)(123)) = \ell_T((12)) = 1 = \ell_T((123)) - \ell_T((13)).$$

We can represent a set partition of $[n]$ pictorially by putting $1, \ldots, n$ clockwise around a circle and drawing convex hulls of the blocks. This is best illustrated by the following example:

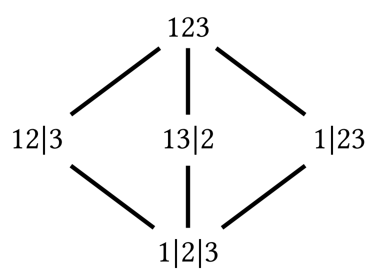
Example 11.3. Let $n = 8$ and consider the partition $\{\{1,3\},\{2,4,5,8\},\{6\},\{7\}\}$.

We say a set partition of $[n]$ is *noncrossing* if none of the blocks cross each other in the picture. The example above is not noncrossing, whereas the one below is noncrossing:

Example 11.4. Again let $n = 8$ and consider the partition $\{\{1,3\},\{2\},\{4,5,8\},\{6,7\}\}$.

Definition 11.5. (Kreweras, 1972) The *noncrossing partition lattice* NC_n is the sublattice of Π_n consisting of the noncrossing partitions.

Example 11.6. For $n = 3$, every partition is noncrossing, and NC_3 looks like:



Fact 11.7. Let c be a Coxeter element of S_n. Then $\mathrm{NC}_n \simeq [e,c]_{ab}$.

Fact 11.8. We have that

$$|\mathrm{NC_n}| = \mathrm{Cat}(S_n) = \frac{1}{n+1}\binom{2n}{n},$$

where $\mathrm{Cat}(S_n)$ is the nth Catalan number.

Definition 11.9. (Brady–Watt, 2002; Bessis, 2003) Let c be a Coxeter element of a finite irreducible Coxeter group W. The c-noncrossing partition lattice is defined to be

$$\mathrm{NC}(W, c) = [e, c]_{ab}.$$

Let $\mathrm{Cat}(W) = |\mathrm{NC}(W, c)|$. This is called the *$W$-Catalan number.*

Theorem 11.10 (Brady–Watt, 2002; Bessis, 2003)**.** *The c-noncrossing partition lattice* $\mathrm{NC}(W, c)$ *is a lattice.*

Take a standard Coxeter element c, and consider the word $c^\infty = ccccc\cdots$. For $w \in W$, the *c-sorting word* is the reduced word for w that is lexicographically first as a subword of c^∞. Let $I_c^{(k)}(w)$ be the set of simple reflections from the kth c in c^∞ used in the c-sorting word for w.

Example 11.11. Let $W = S^4$ and let $c = s_1s_2s_3$. The element $w = 4132$ has c-sorting word $s_2s_3s_2s_1$, where $I_c^{(1)}(w) = \{s_2, s_3\}$, $I_c^{(2)}(w) = \{s_2\}$, and $I_c^{(3)}(w) = \{s_1\}$. Add how to go about doing this later.

The element $w' = 1432$ has c-sorting word $s_2s_3s_2$, and $I_c^{(1)}(w') = \{s_2, s_3\}$ and $I_c^{(2)}(w') = \{s_2\}$.

Definition 11.12. An element $w \in W$ is called c-sortable if $I_C^{(1)}(w) \supset I_c^{(2)}(w) \supset I_c^{(3)}(w) \supset \cdots$.

Exercise 11.13. Let $W = S_n$. Let $c = s_1s_2\cdots s_{n-1}$. Then w is c-sortable if and only if it avoids 312.

Theorem 11.14 (Reading)**.** *Fix a Coxeter element c of W. Then the number of c-sortable elements of W is* $\mathrm{Cat}(W)$.

Proof idea. Let $s_{i_1}, \ldots, s_{i_N}$ be the c-sorting word for w_0. Let $t_j = s_{i_1}\cdots s_{i_j}\cdots s_{i_2}s_{i_1}$. Then $t_1, t_2, \ldots, t_N$ is a list of all reflections of W. For $w \in W$, let $\mathrm{cov}(w) = \{t \in T \mid tw \lessdot_R w\} = \{wsw^{-1} \mid s \in D_R(w)\}$. Let $\psi(w)$ be the element of W obtained by multiplying the reflections in $\mathrm{cov}(w)$ in the order that they appear in the list $t_N, \ldots, t_2, t_1$. Then ψ restricts to a bijection from the set of c-sortable elements to the noncrossing partition lattice $\mathrm{NC}(W, c)$. □

Let $\mathrm{Camb}_c = \{w \in W \mid w \text{ is } c \text{ sortable}\}$. View Camb_c as a subposet of the right weak order.

Theorem 11.15 (Reading)**.** Camb_c *is a sublattice of the right weak order. We call it the c-*Cambrian lattice*. For each $w \in W$, the set* $\{y \in W \mid y \le_R w\} \cap \mathrm{Camb}_c$ *has a unique maximal element, which we denote by $\pi_c^\downarrow(w)$.*

12. Thursday October 12

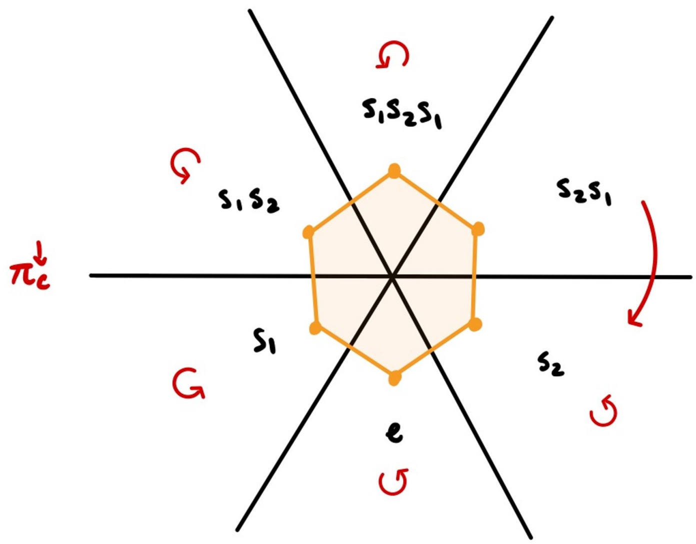
12.1. **More on Cambrian Lattices.** Let W be a finite irreducible Coxeter group. Fix a standard Coxeter element c. Let Camb_c be the set of c-sortable elements of W, viewed as a subposet of the (right) weak order. Recall the following theorem, which was stated originally as Theorem 12.1:

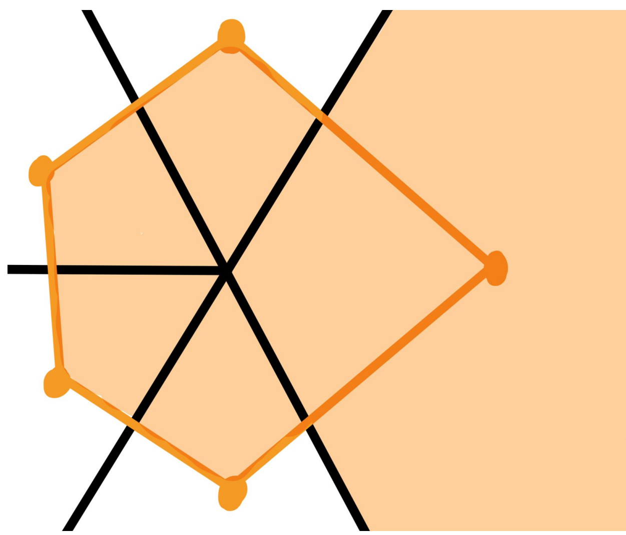
Theorem 12.1 (Reading)**.** Camb_c *is a sublattice of the right weak order. For each $w \in W$, the set* $\{y \in \mathrm{Camb}_c \mid y \le w\}$ *has a unique maximal element $\pi_c^\downarrow(w)$. The map $\pi_c^\downarrow : W \to$* Camb_c *is a surjective lattice homomorphism from the weak order to* Camb_c.

Fact 12.2. When $W = S_n$, the Hasse diagram of Camb_c is isomorphic (as a graph) to the 1-skeleton of an $(n-1)$-dimensional polytope called the *associahedron.*

Define an equivalence relation on W as follows: $u \equiv v$ if and only if $\pi_c^\downarrow(u) = \pi_c^\downarrow(v)$. In S_n, glue the equivalent regions of the braid arrangement.

Example 12.3. Consider $W = S_3$, and let $c = s_1 s_2$. Recall the associated braid arrangement:





When $W = S_n$ and $c = s_1 s_2 \cdots s_{n-1}$, we have that Camb_c is the set of 312-avoiding permutations in S_n, and Camb_c is the nth *Tamari lattice*, which was introduced by Dev Tamari in 1962. In this case, the *bipartite Coxeter elements* are $(s_1 s_3 s_5 \cdots)(s_2 s_4 s_6 \cdots)$ and $(s_2 s_4 s_6 \cdots)(s_1 s_3 s_5 \cdots)$. More generally, if W is finite and irreducible, its Coxeter graph is a tree. Find a bipartition $X \sqcup Y$ of the simple reflections, and let $c_X = \prod_{x \in X} s$ and $c_Y = \prod_{s \in Y} s$. Then $c_X c_Y$ and $c_Y c_X$ are bipartite Coxeter elements.

Example 12.4. Consider D_6.

Theorem 12.5 (Barnard–Defant–Hanson, 2023++)**.** *The maximum number of iterations of* Pop *needed to send an element of* Camb_c *to the bottom element* $\widehat{0}$ *is* $h - 1$, *where* $h = 2|T|/|S|$ *is the Coxeter number of* W.

Theorem 12.6 (Hong, 2022). *Let $a_t(n)$ be the number of elements of the nth Tamari lattice that require t or fewer iterations of* Pop *to reach $\widehat{0}$. Then*

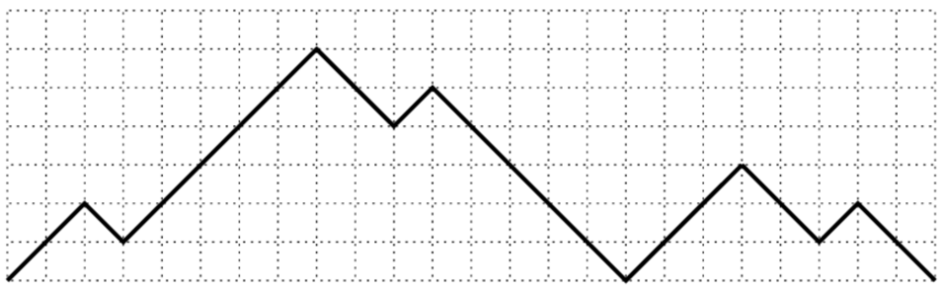
$$\sum_{n\geq} a_t(n)z^n = \frac{z}{1-2z-\sum_{j=2}^{t} C_{j-1}z^j},$$

where here C_{j-1} is the $(j-1)$st Catalan number.

12.2. **Dyck Paths.**

Definition 12.7. A *Dyck path* is a lattice path using unit up steps and unit down steps starting at the origin and ending on the x-axis such that the path never passes below the x-axis. The number of Dyck paths of length $2n$ is C_n.

Example 12.8. Here is an example of a Dyck path:

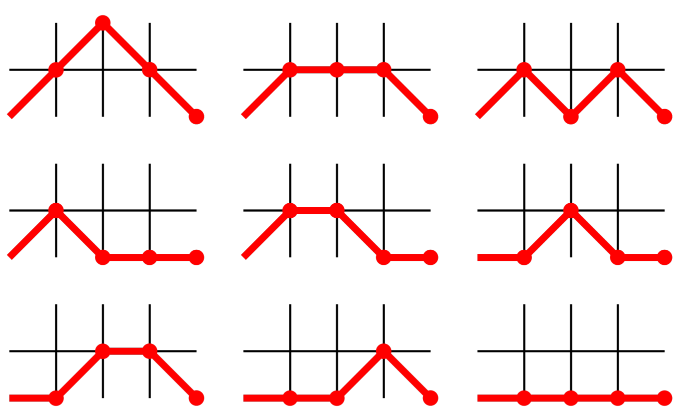


Define a partial order on the set of Dyck paths of length $2n$ so that $\Lambda \leq \Lambda'$ if Λ lies weakly below. We illustrated this partial order using the following example:

Example 12.9. Set $n = 3$. We see that the partial order on the Dyck paths of length 6 is given below. There are 5 of these, since 5 is the third Catalan number. Add picture later.

A *Motzkin path* is a lattice path using unit up, unit down, and unit horizontal steps that likewise begins at the origin, ends on the x-axis, and never passes below the x-axis. The number of Motzkin paths of length n is called the nth *Motzkin number.*

Example 12.10. Below are the Motzkin paths of length 4; hence we see that the 4th Motzkin number is 9.



We can also consider Pop on the lattice of Dyck paths.

Example 12.11. Add picture later.

Theorem 12.12 (Sapoinakis–Tasoulas–Tsikouras, 2006)**.** *The size of the image of* Pop *on the lattice of Dyck paths of length* $2n$ *is*

$$\sum_{k=0}^{n-1} \frac{1}{k+1}\binom{2k}{k}\binom{n+k-1}{3k}.$$[11]

Hint for the following homework problem. Problem Set 3, Problem 1(b): Show that $D_n > n/2 - o(n)$. Hint: every permutation in the image of Pop on S_n has all of its descending runs of size at most 3 (prove this if you want to use it).

Theorem 12.13 (Hong, 2022)**.** *The size of the image of* Pop *on the* n*th Tamari lattice is the* $(n-1)$*st Motzkin number.*

Example 12.14. Add example later.

The Type-B Tamari lattice $\mathrm{Tam}(B_n)$ is Camb_c, where c is the Coxeter element $s_0 s_1 \cdots s_{n-1}$ of B_n.

Theorem 12.15 (Choi–Sun, 2023+)**.** *The size of the image of* Pop *on* $\mathrm{Tam}(B_n)$ *is*

$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-1}{k}\binom{n+1-k}{k}.$$

13. Tuesday October 17

13.1. Cambrian Lattices and Toric Posets.

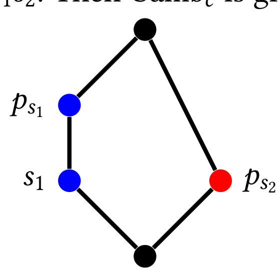
Here are some examples of Cambrian lattices for $W = B_3$. Add pictures later.

Consider $\mathrm{Camb}_c = \{c\text{-sortable elements of } W\}$. The set of *atoms* (i.e., the elements covering $\hat{0}$) are the simple reflections. For $s \in S$, it turns out that

$$\{x \in \mathrm{Camb}_c \mid x \geq s \text{ and } x \not\geq s' \text{ for all } s' \in S \setminus \{s\}\}$$

has a unique maximal element p_s. The p_s's are intimately connected to *quiver representations*. Let $\theta_c = \{x \in \mathrm{Camb}_c \mid x \geq p_s \text{ for some } s \in S\}$.

Example 13.1. Let $W = S_3$ and $c = s_1 s_2$. Then Camb_c is given by the following:



Theorem 13.2 (Barnard–Defant–Hanson, 2023++)**.** *An element* $w \in \mathrm{Camb}_c$ *is in the image of* Pop *if and only if* $w \notin \theta_c$ *and the right descents of* w *all commute with each other.*

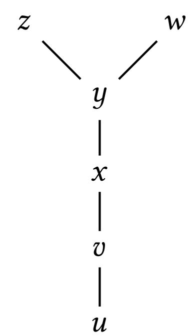
Let G be a graph. An *acyclic orientation* of G is an orientation of the edges of G with no directed cycles. If α is an acyclic orientation, we obtain a partial order $\leq_\alpha$ on the set of vertices in which $u \leq_\alpha v$ if and only if there is a directed path in α from u to v.

Example 13.3. Consider the following graph, whose nodes we replace by vertex labels:

[11] This was refined by Choi and Sun recently.

$$\begin{tikzcd}
u \arrow[r] \arrow[d] & w & y \arrow[l] \arrow[d] \\
v \arrow[r] & x \arrow[u] \arrow[ur] \arrow[r] & z
\end{tikzcd}$$

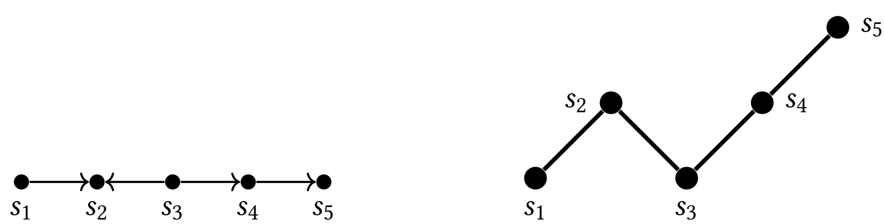
The resulting poset is given by



Let (W, S) be a Coxeter system with finitely many simple reflections. Let Γ be the Coxeter graph.

Proposition 13.4. *The standard Coxeter elements of W correspond bijectively to acyclic orientations of Γ. The reduced words of the standard Coxeter element corresponding to α are the linear extensions of $(S, \leq_\alpha)$.*

Example 13.5. We illustrate the theorem using $W = S_6$. Consider the following acyclic orientation of the Coxeter graph with its corresponding poset



Hence, the reduced words of the standard Coxeter element corresponding to this acyclic orientation are

$$\{s_1s_3s_2s_4s_5,\ s_1s_3s_4s_2s_5,\ s_1s_3s_4s_5s_2,\ s_3s_1s_4s_5s_2,\ s_3s_1s_4s_2s_5,\ s_3s_1s_2s_4s_5\}.$$

Proof. If $s_{i_1}\cdots s_{i_n}$ and $s_{j_1}\cdots s_{j_n}$ are two linear extensions of $(S, \leq_\alpha)$, then we can get from one to the other using Bender–Knuth involutions (by a problem on Problem Set 2). These correspond to commutation moves. Thus $s_{i_1}\cdots s_{i_n} = s_{j_1}\cdots s_{j_n}$ (as elements of W).

Conversely, if $s_{i_1}\cdots s_{i_n} = s_{j_1}\cdots s_{j_n}$ (in W), then by Matsumoto's Theorem, we can get from one to the other by applying commutation moves. These do not affect the acyclic orientation. □

Given some acyclic orientation and a sink in the orientation, we can perform a *source-to-sink* move by reversing the arrows going out of the source. Likewise, we can perform *sink-to-source* moves. For example:

$$\begin{tikzcd} u \arrow[r] \arrow[d] & w & y \arrow[l] \arrow[d] \\ v \arrow[r] & x \arrow[u] \arrow[r] \arrow[ur] & z \end{tikzcd} \qquad \begin{tikzcd} u \arrow[d] & w \arrow[l] \arrow[r] \arrow[d] & y \arrow[d] \\ v \arrow[r] & x \arrow[r] \arrow[ur] & z \end{tikzcd}$$

If s is a source of α, then the corresponding Coxeter element has a reduced word starting with s. Performing a source-to-sink move to s just conjugates the associated Coxeter element by s.

Example 13.6. Let $W = S_6$. The acyclic orientation

$$\underset{s_1}{\bullet} \longrightarrow \underset{s_2}{\bullet} \longleftarrow \underset{s_3}{\bullet} \longrightarrow \underset{s_4}{\bullet} \longrightarrow \underset{s_5}{\bullet}$$

corresponds to $s_3s_1s_2s_4s_5$. Performing a source-to-sink move at s_3, we have the following acyclic orientation

$$\underset{s_1}{\bullet} \longrightarrow \underset{s_2}{\bullet} \longrightarrow \underset{s_3}{\bullet} \longleftarrow \underset{s_4}{\bullet} \longrightarrow \underset{s_5}{\bullet}$$

which corresponds to $s_1s_2s_4s_5s_3$.

A *toric poset* of a graph G is an equivalence class of acylic orientations of G, where two acyclic orientations are equivalent if one can be obtained in the other via source-to-sink and sink-to-source moves.

Theorem 13.7. *Two standard Coxeter elements of W are conjugate if and only if their corresponding acyclic orientations of Γ belong to the same toric poset.*

13.2. **Geometric Representations.** Let (W, S) be a Coxeter system. Let V be a real vector space with basis $\{\alpha_s \mid s \in S\}$. Define a bilinear form $(\bullet|\bullet)$ on V by letting $(\alpha_s|\alpha_{s'}) = -\cos(\pi/m(s, s'))$. For $s \in S$, define $\sigma_s : V \to V$ by $\sigma_s(\beta) = \beta - 2(\alpha_s|\beta)\alpha_s$. The map $s \mapsto \sigma_s$ extends to a faithful representation of W. That is, for $w = s_{i_1} \cdots s_{i_r} \in W$, we can define $\sigma_w = \sigma_{s_{i_1}} \cdots \sigma_{s_{i_r}}$, and the map $w \mapsto \sigma_w$ is a well-defined injective group homomorphism $W \to \mathrm{GL}(V)$. This is called the *standard geometric representation* of W.

Example 13.8. Let $W = S_n$, and let

$$V = \left\{ (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n \;\middle|\; \sum_i \gamma_i = 0 \right\}.$$

Let e_i be the ith standard basis vector in $\mathbb{R}^n$ (i.e., the vector whose ith coordinate is 1 and all other coordinates 0). Let $\alpha_{s_i} = \alpha_i = e_i - e_{i+1}$. It is not difficult to compute that $(\alpha_i|\alpha_i) = -\cos(\pi/1) = 1$ and that $(\alpha_i|\alpha_{i\pm1}) = -\cos(\pi/3) = 1/2$ and that $(\alpha_i|\alpha_j) = -\cos(\pi/2) = 0$ for $|i - j| \geq 2$. Hence,

$$\sigma_{s_i}(\alpha_i) = \alpha_i - 2(\alpha_i|\alpha_i)\alpha_i = -\alpha_i = e_{i+1} - e_i;$$

$$\sigma_{s_i}(\alpha_{i+1}) = \alpha_{i+1} - 2(\alpha_i|\alpha_{i+1})\alpha_i = \alpha_{i+1} + \alpha_i = e_i - e_{i+2};$$

$$\sigma_{s_i}(\alpha_{i-1}) = \alpha_{i-1} - 2(\alpha_i|\alpha_{i-1})\alpha_i = \alpha_{i-1} + \alpha_i = e_{i-1} - e_{i+1}.$$

For $|i - j| \geq 2$, we have $\sigma_{s_i}(\alpha_j) = \alpha_j - 2(\alpha_i|\alpha_j)\alpha_i = \alpha_j$.

14. Thursday October 19

14.1. **The Geometric Representation and Roots.** Let V be a real vector space with basis $\{\alpha_s \mid s \in S\}$. Define bilinear form on V by $(a_s|a_{s'}) = -\cos(\pi/m(s,s'))$. Define $\sigma_s : V \to V$ by $\sigma_s(\beta) = \beta - 2(\alpha_s|\beta)\alpha_s$. For $w = s_{i_1} \cdots s_{i_r}$, we have $\sigma_w = \sigma_{s_{i_1}} \cdots \sigma_{s_{i_r}}$.

Recall the following example from last lecture:

Example 14.1. Let $W = S_n$, and let

$$V = \left\{ (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n \;\middle|\; \sum_i \gamma_i = 0 \right\}.$$

Let e_i be the ith standard basis vector in $\mathbb{R}^n$ (i.e., the vector whose ith coordinate is 1 and all other coordinates 0). Let $\alpha_{s_i} = \alpha_i = e_i - e_{i+1}$. It is not difficult to compute that $(\alpha_i|\alpha_i) = -\cos(\pi/1) = 1$ and that $(\alpha_i|\alpha_{i\pm 1}) = -\cos(\pi/3) = 1/2$ and that $(\alpha_i|\alpha_j) = -\cos(\pi/2) = 0$ for $|i-j| \geq 2$. Hence,

$$\sigma_{s_i}(\alpha_i) = \alpha_i - 2(\alpha_i|\alpha_i)\alpha_i = -\alpha_i = e_{i+1} - e_i;$$

$$\sigma_{s_i}(\alpha_{i+1}) = \alpha_{i+1} - 2(\alpha_i|\alpha_{i+1})\alpha_i = \alpha_{i+1} + \alpha_i = e_i - e_{i+2};$$

$$\sigma_{s_i}(\alpha_{i-1}) = \alpha_{i-1} - 2(\alpha_i|\alpha_{i-1})\alpha_i = \alpha_{i-1} + \alpha_i = e_{i-1} - e_{i+1}.$$

For $|i-j| \geq 2$, we have $\sigma_{s_i}(\alpha_j) = \alpha_j - 2(\alpha_i|\alpha_j)\alpha_i = \alpha_j$. So if $\beta = (\beta_1, \dots, \beta_n)$, then $\sigma_{s_i}(\beta) = (\beta_1, \dots, \beta_{i+1}, \beta_i, \dots, \beta_n)$. In general, $\sigma_w(\beta) = (\beta_{w^{-1}(1)}, \dots, \beta_{w^{-1}(n)})$. From here on out, we will often use $w\beta$ to denote $\sigma_w(\beta)$ for convenience and cleanliness of notation.

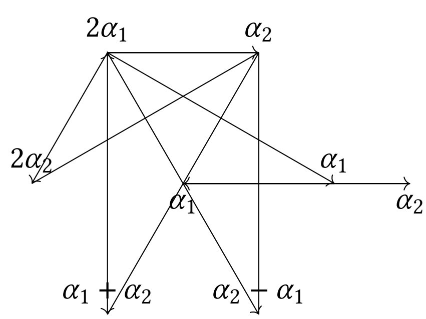
Example 14.2. For an even more concrete example, consider the dihedral group of order 12, $I_2(6)$. We see that

$$s_1\alpha_{s_2} = \alpha_{s_2} - 2(\alpha_{s_2}|\alpha_{s_1})\alpha_{s_1} = \alpha_{s_2} + 2\cos(\pi/6)\alpha_{s_1}$$

and

$$s_1\alpha_{s_1} = \alpha_{s_1} - 2(\alpha_{s_1}|\alpha_{s_1})\alpha_{s_1} = -\alpha_{s_1}.$$

We can visualize the $I_2(m)$-action on this vector space geometrically as reflections of the vectors. Add figure later.



Proposition 14.3. *The action of W on V preserves $(\bullet|\bullet)$.*

Proof. For $\beta, \gamma \in V$ and $s \in S$, we have

$$\begin{aligned}(s\beta|s\gamma) &= (\beta - 2(\alpha_s|\beta)\alpha_2|\gamma - 2(\alpha_s|\gamma)\alpha_s) \\ &= (\beta|\gamma) - 2(\alpha_s|\gamma)(\beta|\alpha_s) - 2(\alpha_s|\beta)(\alpha_s|\gamma) + 4(\alpha_s|\beta)(\alpha_s|\gamma)(\alpha_s|\alpha_s) = (\beta|\gamma)\end{aligned}$$

because $(\beta|\alpha_s) = (\alpha_s|\beta)$ and $(\alpha_s|\alpha_s)$. □

Let $\phi = \{w\alpha_s \mid w \in W, s \in S\}$. We call ϕ the *root system* of W, and elements of ϕ are called *roots*. For $s \in S$, we call α_s *simple roots*.

Fact 14.4. Every root $\beta \in \phi$ can be written uniquely as $\sum_{s\in S} c_s\alpha_s$ for some real numbers c_s. Moreover, the coefficients c_s are either all nonnegative or all nonpositive.

A root is said to be *positive* if it is in $\mathbb{R}_{\geq 0}\text{span}\{\alpha_s\}$; a root is said to be *negative* if it is in $\mathbb{R}_{\leq 0}\text{span}\{\alpha_s\}$. Let ϕ^+ and ϕ^- denote the sets of positive and negative roots, respectively. Note that $-\phi^- = \phi^+$. Moreover, we may write

$$\phi = \phi^+ \sqcup \phi^-.$$

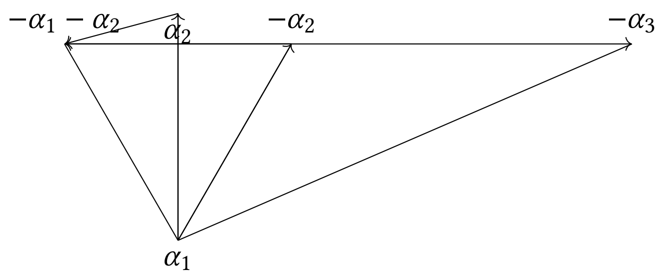
Since the action of W preserves $(\bullet|\bullet)$ and $(\alpha_s|\alpha_s) = 1$ for all $s \in S$, it follows that $(\beta|\beta) = 1$ for all roots $\beta \in \phi$. Thus, $\mathbb{R}\beta \cap \phi = \{\pm\beta\}$.

Recall that for a vector space V, its dual V^* is the space of linear functions $p : V \to \mathbb{R}$. There is a natural pairing of V^* and V defined by $\langle p, \beta\rangle = p(\beta)$. For $\beta \in V$, consider the hyperplane

$$H_\beta = \{p \in V^* \mid p(\beta) = 0\}.$$

The *Coxeter arrangement* of W is the set $\mathcal{H}_W = \{H_\beta \mid \beta \in \phi\}$.

Example 14.5. Let $W = S_3$ and $V = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3 \mid \gamma_1 + \gamma_2 + \gamma_3 = 0\}$. Fix pictures (ChatGPT made these).



Example 14.6. Let $W = S_n$ and V the standard geometric representation of W. Then

$$\phi = \{e_i - e_j \mid 1 \leq i, j \leq n, i \neq j\},$$

and $w(e_i - e_{i+1}) = e_{w(i)} - e_{w(i+1)}$. We have

$$\phi^+ = \{e_i - e_j \mid 1 \leq i \leq j \leq n\} \quad \text{and} \quad \phi^- = \{e_j - e_i \mid 1 \leq i \leq j \leq n\}.$$

We may think of V^* as $\mathbb{R}^n/\text{span}\{(1, \ldots, 1)\}$, where $\langle (x_i), (\gamma_i)\rangle = \sum_i x_i\gamma_i$. Then $H_{e_i - e_j} = \{(x_1, \ldots, x_n) \in V^* \mid x_i = x_j\}$.

Example 14.7. Let $W = B_n$ and V the standard geometric represntation of W. Note that $\alpha_{s_i} = \alpha_i = e_i - e_{i+1}$ if $1 \leq i \leq n-1$ and that $\alpha_{s_0} = \alpha_0 = e_1$. Moreover, we have

$$\phi = \{\langle,\rangle me_i \pm e_j \mid 1 \leq i \leq j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\};$$
$$\phi^+ = \{e_i \pm e_j \mid 1 \leq i \leq n\} \cup \{e_i \mid 1 \leq i \leq n\};$$
$$\phi^- = \{-e_i \pm e_j \mid 1 \leq i \leq n\} \cup \{-e_i \mid 1 \leq i \leq n\}.$$

Regard V^* as $\mathbb{R}^n$, where $\langle (x_i), (\gamma_i)\rangle = \sum_i x_i\gamma_i$. Then $H_{e_i - e_j} = \{(x_1, \ldots, x_n) \mid x_i = x_j\}$, $H_{e_i + e_j} = \{(x_1, \ldots, x_n) \mid x_i = -x_j\}$, and $H_{e_i} = \{(x_1, \ldots, x_n) \mid x_i = 0\}$.

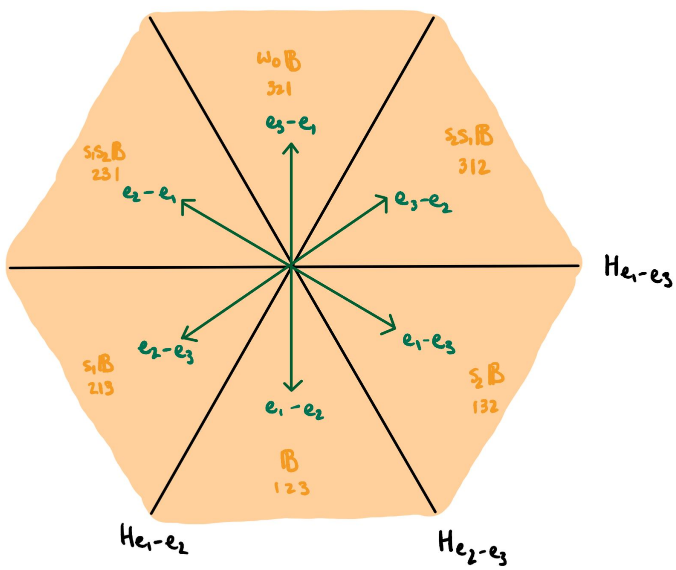
A *region* of the Coxeter arrangement of $\mathcal{H}_W$ is the closure of a connected component of $V^* \setminus \bigcup_{\beta\in\phi} H_\beta$. Let $\mathbb{B} = \{p \in V^* \mid p(\alpha_s) \geq 0 \text{ for all } s \in S\}$. We call $\mathbb{B}$ the *base region*.

There is an action of W on V^* satisfying $\langle wp, \beta\rangle = \langle p, w^{-1}\beta\rangle$ for all $p \in V^*$, $\beta \in V$, and $w \in W$. This induces an action of W on the regions of $\mathcal{H}_W$. This is a *free* action (i.e., for any region R, the only element of W that fixes R is e). This allows us to identify $w \in W$ with $w\mathbb{B}$.

15. Tuesday October 24

15.1. **The Tits Cone.** Recall that V is a real vector space with basis $\{\alpha_s \mid s \in S\}$; let V^* denote the dual space. For $p \in V^*$ and $\beta \in V$, we let $\langle p, \beta\rangle = p(\beta)$. For $s \in S$ and $\beta \in V$, we have $s\beta = \beta - 2(\alpha_s|\beta)\alpha_s$. This extends to an action of W on V. The *root system* $\phi = \{w\alpha_s \mid s \in S, w \in W\}$. Let $H_\beta = \{p \in V^* \mid p(\beta) = 0\}$. Also recall that $\mathcal{H}_\beta = \{H_\beta \mid \beta \in \phi\}$ is called the *Coxeter arrangement* of W. There is an action of W on V^* given by $\langle wp, \beta\rangle = \langle p, w^{-1}\beta\rangle$. A *region* of $\mathcal{H}_W$ is the closure of a connected component of $V^* \setminus \bigcup_{\beta\in\phi} H_\beta$. We obtain a free action of W on the set of regions of $\mathcal{H}_W$. The *base region* is $\mathbb{B} = \{p \in V^* \mid p(\alpha_s) \geq 0 \text{ for all } s \in S\}$. We can identify $w \in W$ with $w\mathbb{B}$.

Example 15.1. Consider $W = S_3$ so that $V = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3 \mid \gamma_1 + \gamma_2 + \gamma_3 = 0\}$.



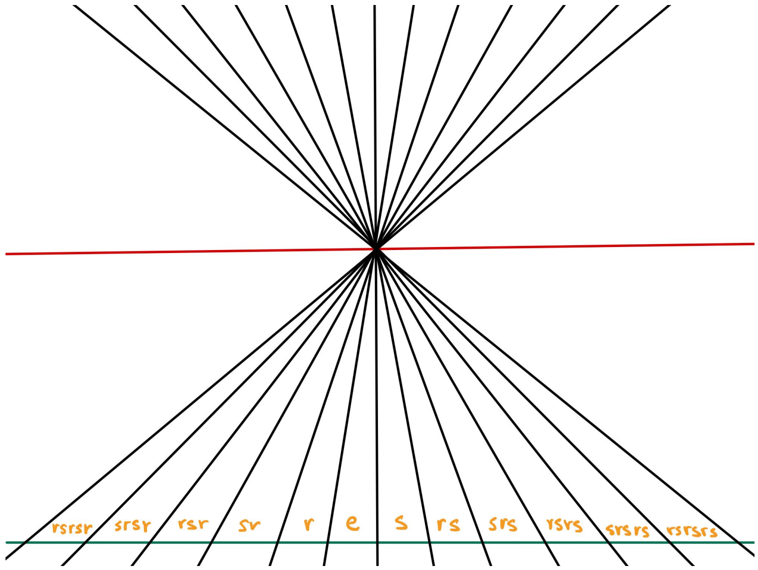
Recall that a bilinear form is positive definite if $B(v, v) > 0$ for all $v \neq 0$.

Remark 15.2. We see that W is finite if and only if $(\bullet|\bullet)$ is positive definite.

When W is finite, every region of $\mathcal{H}_W$ is identifies with some unique element of W. This is false when W is infinite. We define the *Tits cone* of W to be

$$\bigcup_{w\in W} w\mathbb{B}.$$

Example 15.3. Consider $I_2(\infty)$, generated by r, s with $m(r, s) = \infty$.



The Tits cone is the open lower half-plane together with the origin.

Example 15.4. Consider $W = \tilde{S}_3$. Add picture later.

Note here that the reduced words for $w \in W$ correspond to maximal chains of $[e, w]_L$.

For $\beta \in \phi$ define $t_\beta : V \longrightarrow V$ by $t_\beta \gamma = \gamma - 2(\gamma|\beta)\beta$ (note that $t_\beta = t_{-\beta}$). If $\beta = \alpha_s$, then $t_\beta = s$. More generally, if $\beta = w\alpha_s$, then

$$t_\beta \gamma = \gamma - 2(\gamma|w\alpha_s)w\alpha_s = w(w^{-1}\gamma - 2(\gamma|2\alpha_s)\alpha_s) = w(w^{-1}\gamma - 2(w^{-1}\gamma|\alpha_s)\alpha_s) = w(sw^{-1}\gamma) = wsw^{-1}\gamma.$$

This shows that t_β and wsw^{-1} agree as elements of $GL(V)$. The standard geometric representation is faithful, so we can identify t_β with wsw^{-1}. Letting $T = \{wsw^{-1} \mid s \in S,\ w \in W\}$, we obtain a map $\rho : \phi^+ \longrightarrow T$ given by $\rho(\beta) = t_\beta$.

Proposition 15.5. *The map $\rho : \phi^+ \longrightarrow T$ is a bijection.*

Proof. To see injectivity, suppose that $\beta, \beta' \in \phi^+$ and $t_\beta = t_{\beta'}$. Then $t_\beta \beta = \beta - 2(\beta|\beta)\beta = \beta - 2\beta = -\beta$. Similarly, we have $t_{\beta'} = \beta - 2(\beta|\beta')\beta'$. Thus, $\beta - 2(\beta|\beta')\beta' = -\beta$, implying $\beta = (\beta|\beta')\beta'$. □

16. Thursday October 26

Recall: $\tilde{S}_3$ has Coxeter diagram a triangle with vertices s_0, s_1, s_2.

Let $T = \{wsw^{-1} : w \in W, s \in S\}$. We obtain a bijection $\rho : \phi^+ \longrightarrow T$ given by $\rho(\beta) = t_\beta$, where $t_\beta \gamma = \gamma - 2(\gamma|\beta)\beta$.

Fact 16.1. For $w \in W$ and and $\gamma \in \phi^+$, $w\gamma \in \phi^-$ iff $t_\gamma \in T_R(w)$.

Example 16.2. $W = S_n$. $\gamma = e_i - e_j$ $(1 \leq i < j \leq n)$. In this case, the reflection $t_\gamma = (i\ j)$. We have $w\gamma = e_{w(i)} - e_{w(j)}$, so $w\gamma \in \phi^-$ iff $w(i) > w(j)$ iff $(i\ j) \in T_R(w)$.

For $X \subseteq W$, let $X\mathbb{B} = \cup_{x \in X} x\mathbb{B}$. Say X is convex [12] if $X\mathbb{B}$ is an intersection of half-spaces determined by hyperplanes in the Coxeter arrangement $\mathcal{H}_W$ and the Tits cone. (If $X = W$, we get the whole Tits cone.)

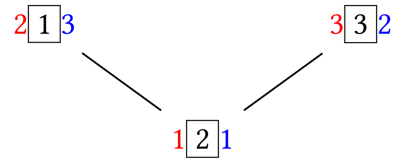
(picture with triangles highlighted $s_2, e, s_1, s_0s_1, s_0, s_1s_0, s_1s_0s_1$).

(diagram with 6 regions 123 132 231 321 312 213 clockwise from bottom, 213 312 shaded).

[12] This definition of convex coincides with the usual geometric notion.

Suppose $P = ([n], \leq_P)$ where $[n] := \{1, 2, \dots, n\}$ is a poset. A linear extension of P is a bijection $w : P \to [n]$ such that $w(i) \leq w(j)$ whenever $i \leq_P j$.

Example 16.3. The linear extensions of



are 213 and 312.

This poset has relations $2 \leq_P 1$ and $2 \leq_P 3$, which correspond to the linear inequalities $x_2 \leq x_1$ and $x_2 \leq x_3$.

The inequality $x_2 \leq x_1$ corresponds to the half-space where the second coordinate is smaller than the first (top side of line with angle 120 from positive x-axis) and the other inequality corresponds to the half-space where the second coordinate is smaller than the third (bottom side of line with angle 60 from positive x-axis).

In general, nonempty convex sets in S_n are in bijective correspondence with partial orders on $[n]$. The elements of the convex set are the linear extensions of the poset. Thus, nonempty convex sets in Coxeter groups generalize finite posets.

(shaded pink region in LHS of board) make diagr

Definition 16.4. Let $\mathcal{L}$ be a convex subset of W. For $s \in S$, define the Bender-Knuth involution $\mathrm{BK}_s : \mathcal{L} \to \mathcal{L}$ by

$$\mathrm{BK}_S(w) = \begin{cases} sw & \text{if } sw \in \mathcal{L}; \\ w & \text{if } sw \notin \mathcal{L}. \end{cases}$$

Fact 16.5. This is a good fact. Any element of $\mathcal{L}$ can be obtained from any other element of $\mathcal{L}$ by applying some sequence of Bender-Knuth involutions.

(diagram modified: s1s0s1 to s1so by BK s0 and 101 to 01 by BKs1) insert diagr

Proof. Convex sets are connected. □

We will take a detour into hyperplane arrangements.

Definition 16.6. A hyperplane in $\mathbb{R}^n$ is a codimension 1 affine subspace. [So a hyperplane is of the form $\{x \in \mathbb{R}^n : f(x) = a\}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a linear functional and $a \in \mathbb{R}$.]

Definition 16.7. A hyperplane arrangement in $\mathbb{R}^n$ is a set of hyperplanes.

Definition 16.8. A hyperplane arrangement $\mathcal{H}$ is central if all of the hyperplanes in $\mathcal{H}$ contain the origin. Instead of affine subspaces, they're linear subspaces.

Assume $\mathcal{H}$ is finite and central.

Definition 16.9. A region of $\mathcal{H}$ is the closure of a connected component of $\mathbb{R}^n \setminus \cup_{H \in \mathcal{H}} H$.

Definition 16.10. A wall of a region R is a hyperplane in $\mathcal{H}$ whose intersection with R has codimension 1.

Definition 16.11. A region is simplicial if it has exactly n walls. Say $\mathcal{H}$ is simplicial if all of its regions are simplicial.

Figure 1. Happy Halloween!

Example 16.12. Every finite central arrangement in $\mathbb{R}^2$ is simplicial.

Exercise on final pset: Assume $\cap_{H\in\mathcal{H}} H = \{0\}$. Find a finite central hyperplane arrangement in $\mathbb{R}^3$ that is not simplicial.

Example 16.13. Let W be a finite Coxeter group. Then $\mathcal{H}_W$ is simplicial. The walls of $w\mathbb{B}$ are the hyperplanes $H_{w^{-1}\alpha_s}$ for $s \in S$.

Definition 16.14. A hyperplane arrangment is essential if $\cap_{H\in\mathcal{H}} H = \{0\}$.

Let $\mathcal{H}$ be a finite central essential hyperplane arrangement in $\mathbb{R}^n$. Fix a base region $\mathbb{B}$ of $\mathcal{H}$. For each region R, let $\mathrm{Inv}(R)$ be the set of hyperplanes in $\mathcal{H}$ that separate R from $\mathbb{B}$.

e diagram

example with R,B invR

Definition 16.15. The poset of regions of $\mathcal{H}$ is the partial order on the set of regions in which $R \subseteq R'$ if $\mathrm{Inv}(R) \subset \mathrm{Inv}(R')$. Note this depends on the choice of base region $\mathbb{B}$.

The poset of regions is the left weak order in the case of Coxeter groups. Consider $\mathcal{H}_W$ as before. For $w \in W$,

$$\mathrm{Inv}(w\mathbb{B}) = \{H_\beta : t_\beta \in T_R(w)\}.$$

Thus, the poset of regions of $\mathcal{H}_W$ is the left weak order on W [since $u \leq_L v$ iff $T_R(u) \subseteq T_R(v)$].

Theorem 16.16 (Bjorner-Edelman-Ziegler)**.** *Let $\mathcal{H}$ be a finite central essential hyperplane arrangement in $\mathbb{R}^n$, and fix a base region $\mathbb{B}$. If $\mathcal{H}$ is simplicial, then the poset of regions is a lattice. True for any choice of base region.*

Summary: Consider the sets T of reflections, ϕ^+ of positive roots, $\mathcal{H}_W$ of hyperplanes in Coxeter arrangments. These are in bijection with one another.

Suppose $\beta \in \phi^+$. For $w \in W$,

$$t_\beta \in T_R(W) \iff w\beta \in \phi^- \iff H_\beta \in \mathrm{Inv}(w\mathbb{B}).$$

17. Halloween

For $\beta \in \phi^+$, let

$$dp(\beta) = \min\{k \mid w\beta \in \phi^- \text{ for some } w \in W, \ell(w) = k\}.$$

We call $dp(\beta)$ the *depth* of β. For example, $dp(\beta) = 1$ if and only if $\beta \in \{\alpha_s \mid s \in S\}$. The *root poset* is the partial order on ϕ^+ defined so that $\beta \lessdot \gamma$ if $dp(\gamma) = dp(\beta) + 1$ and $\gamma = s\beta$ for some $s \in S$. The minimal elements of the root poset are the simple roots. The root poset is graded with rank function dp.

Example 17.1. Consider $S_5 = A_4$. Write $abcd$ for $a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4$. Add picture later.

Fact 17.2. Order ideals of the root poset of type A_{n-1} are in bijection with Dyck paths of size $2n$. Add picture later.

Use $\phi^+(W)$ to denote the root poset of W. By the fact above, we have $|J(\phi^+(A_{n-1}))|$ is the Catalan number $\mathrm{Cat}(S_n) = \mathrm{Cat}(A_{n-1})$.

Fact 17.3. If W is a finite irreducible Coxeter group, then $|J(\phi^+(W))| = \mathrm{Cat}(W)$.

Example 17.4. Consider $\tilde{S}_3$. Write $\boxed{abc}$ for $a\alpha_1 + b\alpha_2 + c\alpha_3$.

17.1. **Enumeration.** For $A \subset W$, let $A(q) = \sum_{w \in A} q^{\ell(w)}$ (assume $|S| < \infty$). We want to compute $W(q)$.

Lemma 17.5. *If $W = W_1 \times \cdots \times W_k$ is the direct product of Coxeter systems, then*

$$W(q) = \prod_{i=1}^{k} W_i(q).$$

Proof. If $w = (w_1, \ldots, w_k)$, where $w_i \in W_i$, then $\ell(w) = \sum_{i=1}^{k} \ell(w_i)$. So

$$W(q) = \sum_{w \in W} q^{\ell(w)} = \sum_{w_1 \in W_1} \cdots \sum_{w_k \in W_k} q^{\ell(w_1) + \cdots + \ell(w_k)} = \prod_{i=1}^{k} W_i(q),$$

as desired. □

Lemma 17.6. *For $J \subset S$, we have $W(q) = W^J(q) W_J(q)$.*

Proof. We have

$$W(q) = \sum_{w \in W} q^{\ell(w)} = \sum_{w \in W} q^{\ell(w_J) + \ell(w^J)} = \sum_{w_J \in W_J} q^{\ell(w_J)} \sum_{w^J \in W^J} q^{\ell(w^J)},$$

as desired. □

Thus, in order to compute $W(q)$, we just need to compute $W_J(q)$ and $W^J(q)$ for some $\emptyset \neq J \subsetneq S$. Since W_J is a Coxeter group, we can assume inductively that we have computed $W_J(q)$. But W^J is not a Coxeter group.

Recall that $D_I^J = \{w \in W \mid I \subset D_R(w) \subset J_S\}$, $D_I = DI^I$, and $W^J = D_\emptyset^{S \setminus J}$.

Proposition 17.7. *For $I \subset J \subset S$, we have*

$$D_I^J(q) = \sum_{J \setminus I \subset K \subset J} (-1)^{|J \setminus K|} W^{S \setminus K}(q).$$

Proof. For each $K \subset S$, we have

$$S^{S \subset K}(q) = \sum_{L \subset K} D_L(q).$$

Thus,

$$\sum_{J \setminus I \subset K \subset J} (-1)^{|J \setminus K|} W^{S \setminus K}(q) = \sum_{J \setminus I \subset K \subset J} (-1)^{|J \setminus K|} \sum_{L \subset K} D_L(q) = \sum_{L \subset J} D_L(q) \sum_{(J \setminus I) \cup L \subset K \subset J} (-1)^{|J \setminus K|}.$$

By the Principle of Inclusion-Exclusion, we have

$$\sum_{(J \setminus I \cup L \subset K \subset J} (-1)^{|J \setminus K|} = \begin{cases} 1 & \text{if } (J \setminus I) \cup L = J; \\ 0 & \text{otherwise.} \end{cases}$$

So,

$$\sum_{L \subset J} D_L(q) \sum_{(J \setminus I) \cup L \subset K \subset J} (-1)^{|J \setminus K|} = \sum_{I \subset L \subset J} D_L(q) = D_I^J(q),$$

as desired. □

Corollary 17.8. *If W is finite, then*

$$\sum_{K\subset S}\frac{(-1)^{|K|}}{W_K(q)}=\frac{q^{\ell(w_0)}}{W(q)}.$$

If W is infinite, then

$$\sum_{K\subset S}\frac{(-1)^{|K|}}{W_K(q)}=0.$$

Proof. Put $I=J=S$ in the Proposition:

$$\frac{D_S^S(q)}{W(q)}=\sum_{K\subset S}(-1)^{|S\setminus K|}\frac{W^{S\setminus K}(q)}{W(q)}=\sum_{K\subset S}\frac{(-1)^{|S\setminus K|}}{W_{S\setminus K}(w)}=\sum_{K\subset S}\frac{(-1)^{|K|}}{W_K(q)}.$$

For W finite, note that $D_S^S(q)=q^{\ell(w_0)}$; for W infinite, note that $D_S^S(q)=0$. □

Example 17.9. Let $W=B_2$ with simple reflections $S=\{a,b\}$ and $m(a,b)=4$. We have $\ell(w_0)=\ell(abab)=4$, so

$$\frac{q^4}{W(q)}=\frac{1}{W_\emptyset(q)}-\frac{1}{W_a(q)}-\frac{1}{W_b(q)}+\frac{1}{W(q)}=1-\frac{1}{1+q}-\frac{1}{q+1}+\frac{1}{W(q)},$$

so

$$W(q)=\frac{1+q}{q^4(q-1)}.$$

Example 17.10. Consider W given by $S=\{a,b,c\}$ with unlabeled edges between b,c, and c,a and an edge with label 4 between a,b. The group W is infinite, so

$$0=\frac{1}{W_\emptyset(q)}-\frac{1}{W_a(q)}-\frac{1}{W_b(q)}-\frac{1}{W_c(q)}+\frac{1}{W_{ab}(q)}+\frac{1}{W_{ac}(q)}+\frac{1}{W_{bc}(q)}-\frac{1}{W(q)}.$$

This is

$$1-\frac{1}{1+q}-\frac{1}{1+q}-\frac{1}{1+q}+\frac{q^4(q-1)}{1+q}+\frac{1}{1+2q+2q^2+q^3}+\frac{1}{1+2q+2q^2+q^3}-\frac{1}{W(q)}.$$

Solving this yields

$$W(q)=\frac{(1+q)(1+q+q^2)(1+q+q^2+q^3)}{1-q^2-q^3-q^4+q^6}.$$

Recall that if $W=A_n$, then $W(q)=\prod_{i=1}^n\frac{1-q^{i+1}}{1-q}=\prod_{i=1}^n[i+1]_q$, where

$$[k]_q=\frac{1-q^k}{1-q}=1+q+q^2+\cdots+q^{k-1}$$

is the q-analogue of the number q.

18. Thursday November 2

Erratum: Maximal elements in the absolute order need not be Coxeter elements.

18.1. **It's Möbin Time.** Let

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^{k-1}.$$

Theorem 18.1. *Suppose (W,S) is finite and irreducible. Let $n = |S|$. There exist positive integers $e_1, \ldots, e_n$ such that*

$$W(q) = \prod_{i=1}^{n} [e_i + 1]_q.$$

In particular, $|W| = \prod_{i=1}^{n}(e_i + 1)$ and $|T| = \ell(w_0) = \sum_{i=1}^{n} e_i$.

Example 18.2. If $W = S_{n+1} = A_n$, then $W(q) = \prod_{i=1}^{n}[i+1]_q$. The numbers $e_1, \ldots, e_n$ are called the exponents of W.

Let (W,S) be an affine irreducible Coxeter system. Let $e_1, \ldots, e_n$ be the exponents of the associated finite Coxeter group. Then

$$W(q) = \prod_{i=1}^{n} \frac{[e_i+1]_q}{1-q^{e_i}}.$$

Example 18.3. We have

$$\widetilde{S_n}(q) = \prod_{i=1}^{n} \frac{[i+1]_q}{1-q^i} = \frac{1-q^{n+1}}{(1-q)^{n+1}}.$$

Let P be a finite poset. The Möbius function of P is the map

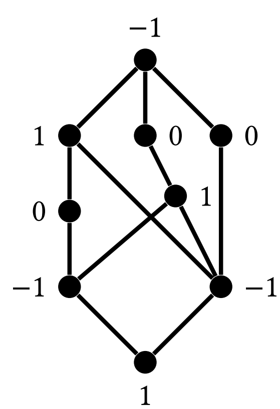
$$\mu = \mu_P : \{(x,y) \in P \times P \mid x \le y\} \to \mathbb{Z}$$

defined by the conditions $\mu(x,x) = 1$ and

$$\sum_{z \in [x,y]} \mu(x,z) = 0$$

for all $x < y$.

Example 18.4. Consider the following poset; we label x with $\mu(\widehat{0}, x)$, where $\widehat{0}$ is the minimal element:



We also note that

$$\mu(x,y) = -\sum_{x \le z < y} \mu(x,z).$$

Exercise 18.5. Show the following.

(1) If P is the nth Boolean lattice, then $\mu(\emptyset, X) = (-1)^{|X|}$.
(2) If P is the lattice of divisors of N, then

$$\mu(1,k) = \begin{cases} (-1)^m & \text{if } k \text{ is a product of } m \text{ distinct primes;} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 18.6 (Möbius Inversion Formula)**.** *Let P be a finite poset with a unique minimal element $\widehat{0}$. Suppose $f, g : P \to \mathbb{C}$ are such that*

$$g(y) = \sum_{x \leq y} f(x) \tag{3}$$

for all $y \in P$. Then

$$f(y) = \sum_{x \leq y} g(x)\mu(x,y) \tag{4}$$

for all $y \in P$.

Proof. Let M and M' be the matrices with rows and columns indexed by P, wehere

$$M_{xy} = \begin{cases} 1 & \text{if } x \leq y; \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad M'_{yx} = \begin{cases} \mu(x,y) & \text{if } x \leq y; \\ 0 & \text{otherwise.} \end{cases}$$

Think of f and g as vectors whose coordinates are indexed by P. Then (3) gives us $g = Mf$, and (4) tells us that $f = M'g$. Thus, it suffices to show that $M' = M^{-1}$. Now,

$$(MM')_{xy} = \sum_{z \in P} M_{xz} M'_{zy} = \sum_{z \leq x} M'_{zy} = \sum_{y \leq z \leq x} \mu(y,z) = \begin{cases} 1 & \text{if } x = y; \\ 0 & \text{otherwise,} \end{cases}$$

and this completes the proof. □

18.2. **The Nerve.** An *abstract simplicial complex* is a collection $\mathcal{F}$ of sets such that if $A \in \mathcal{F}$, then every subset of A is in $\mathcal{F}$.

Suppose W is infinite. The *nerve* of (W,S) is $\mathcal{N}(W,S) = \{J \subset S \mid \#W_J < \infty\}$. Note that $\mathcal{N}(W,S)$ is an abstract simplicial complex. We may think of $\mathcal{N}(W,S) \cup \{S\}$ as a poset under inclusion.

Example 18.7. Consider the Coxeter group W generated by a, b, c, d whose Coxeter graph has an edge labeled 4 between a and b, an unlabeled edge between a and c, an edge labeled 8 between b and c, and an unlabeled edge between c and d. We see that

$$\mathcal{N}(W,S) = \{\emptyset, a, b, c, d, ab, bc, ac, ad, bd, cd, abd, acd\}.$$

Let $\mu_{\mathcal{N}}$ be the Möbius function of $\mathcal{N}(W,S) \cup \{S\}$. If $I, K \in \mathcal{N}(W,S)$ and $K \subset I$, then $[K, I]$ is isomorphic to the Boolean lattice of subsets of $I \setminus K$ (note that $I \setminus K \supset J \mapsto K \cup J$). Hence, $\mu_{\mathcal{N}}(K,I) = (-1)^{|I \setminus K|}$.

Proposition 18.8. *If (W,S) is an infinite Coxeter system, then*

$$\frac{1}{W(q)} = -\sum_{K \subset \mathcal{N}(W,S)} \frac{\mathcal{N}(K,S)}{W_k(q)}.$$

Proof. Let $\mathcal{N} = \mathcal{N}(W, S)$. If $I \subset S$ and $I \notin \mathcal{N}$, then W_I is infinite, so

$$\sum_{K \subset I} \frac{(-1)^{|I \setminus K|}}{W_K(q)} = 0.$$

Then

$$\sum_{I \notin \mathcal{N}} \sum_{K \subset I} \frac{(-1)^{|I \setminus K|}}{W_K(q)} = 0.$$

Hence,

$$\sum_{K \subset S} \frac{1}{W_K(q)} \sum_{I \supset K, I \notin \mathcal{N}} (-1)^{|I \setminus K|} = 0.$$

For fixed K,

$$\sum_{I \supset K, I \notin \mathcal{N}} (-1)^{|I \setminus K|} = -\sum_{I \supset K, I \notin \mathcal{N}} (-1)^{|I \setminus K|} = -\sum_{I \supset K, I \in \mathcal{N}} \mu_{\mathcal{N}}(K, I) = \mu_{\mathcal{N}}(K, S),$$

as desired. ◻

19. Tuesday November 7

Recall that if (W, S) is an infinite Coxeter system, then

$$-\frac{1}{W(q)} = \sum_{K \in \mathcal{N}(W,S)} \frac{\mu_{\mathcal{N}}(K, S)}{W_K(q)}$$

where $\mu_{\mathcal{N}}$ is the Möbius function of $\mathcal{N}(W, S) \cup \{S\}$.

Example 19.1. Let $\mathcal{U}_n$ be the *universal Coxeter group* on n generators. The Coxeter graph of $\mathcal{U}_n$ is a complete graph with n vertices and each edge labeled ∞. Then $\mathcal{N}(\mathcal{U}_n, S) = \{J \subset S \mid |J| \leq 1\}$. We have that $\mathcal{N}(\mathcal{U}_n, S) \cup \{S\}$ is given by

$$\begin{tikzcd}
 & & S \arrow[dll, dash] \arrow[dl, dash] \arrow[dr, dash] & \\
\{s_1\} & \{s_2\} & \cdots & \{s_n\} \\
 & & \varnothing \arrow[ull, dash] \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

We see that $\mu_{\mathcal{N}}(S, S) = 1$, $\mu_{\mathcal{N}}(\{s_i\}, S) = -1$, and $\mu_{\mathcal{N}}(\varnothing, S) = n - 1$. So

$$-\frac{1}{\mathcal{U}_n(q)} = n - 1 + n \cdot \frac{(-1)}{1 + q} = \frac{(n-1)q - 1}{1 + q}.$$

19.1. Tableaux.

A *partition* of n is a tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$. The numbers $\lambda_1, \dots, \lambda_k$ are the *parts* of λ. The *length* of λ is k. We can represent a partition as a *Young diagram*—a collection of left-justified rows of boxes such that the ith row has λ_i boxes. Young diagrams can be viewed as posets by rotating the diagram 45° clockwise and viewing the resulting figure as a Hasse diagram whose vertices are the boxes with edges given by box adjacencies.

Example 19.2. The partition $(5, 4, 2, 2, 1)$ of 13 corresponds to the following Young diagram:

A *standard Young tableau* of shape λ is a filling of the boxes λ with $1, \ldots, n$ so that the rows and columns are increasing.

Example 19.3. The standard Young tableau of shape $(3, 3)$ are

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6

If we view a young diagram λ as a poset, then the standard Young tableaux correspond to linear extensions of the poset.

Flipping a standard Young tableau T of shape λ across the main diagonal yields a standard Young tableau T' of shape λ'. Call λ' and T' the *transpose* of λ and T, respectively. We use f^λ to denote the number of standard Young tableaux of shape λ.

For a box b in a Young diagram λ, its hook length, denoted h_b, is the number of boxes strictly below or to the right of b plus 1.

Example 19.4. The boxes below or to the right of b are labeled with an X:

	b	X		
	X			
	X			

Hence, the hook length of b is 4.

Theorem 19.5 (The Hook-Length Formula)**.** *Let λ be a partition of n. Then*

$$f^\lambda = \frac{n!}{\prod_{b \in \lambda} h_b}$$

Example 19.6. For the partition $(4, 2, 1)$, the hook lengths of each box are given by

6	4	2	1
3	1		
1			

,

so

$$f^\lambda = \frac{7!}{6 \cdot 4 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 35.$$

19.2. **A Representation-theoretic Interlude.**

Fact 19.7. Partitions of n correspond to irreducible representations of S_n. The dimension of the irreducible representation corresponding to λ is f^λ.

Example 19.8. Consider S_3. The partition (3) corresponds to the trivial representation—the one-dimensional representation given by $w \cdot \beta = \beta$ for all $w \in S_3$. The partition $(1, 1, 1)$ corresponds to the sign representation—the one-dimensional representation given by $w \cdot \beta = (-1)^{\ell(w)}\beta$. Finally, the partition $(2, 1)$ corresponds to the standard geometric representation, which, recall, is two-dimensional.

In general, if G is a finite group, then we have

$$|G| = \sum_{\text{irreps } V \text{ of } G} \dim(V)^2.$$

When $G = S_n$, this tells us that

$$n! = \sum_{\lambda \in \pi(n)} (f^\lambda)^2,$$

where $\pi(n)$ denotes the partitions of n.[13] A bijective proof of this fact is given by the Robinson–Schensted correspondence (commonly referred to as the RS or RSK correspondence, or even as RS or RSK).[14] The RS correspondence is a bijection

$$\mathrm{RS} : S_n \longrightarrow \bigcup_{\lambda \in \pi(n)} (SYT(\lambda) \times SYT(\lambda)),$$

where $SYT(\lambda)$ denotes the set of standard Young tableaux of shape λ. We illustrate RS with the following example.

Example 19.9. Let $n = 5$, and let $w = 25134$.

[13] One way of proving this formula is via the Robinson–Schensted correspondence. Another way, which I prefer, was given by Vershik and Okounkov in this paper. In the paper, they show that standard Young tableaux of shape λ index basis vectors of the representation S_n corresponding to λ. From this, the formula is obvious.

[14] Craige Schensted now goes by the name Ea Ea. He changed his name to Ea, the Babylonian name for the Sumerian god Enki, in 1995. In 1999, he changed his name again, this time to Ea Ea. Thus, arguably, the RS (or RSK) correspondence, as it is commonly referred to, should be renamed the RE (or REK) correspondence. See here for a more detailed account of Ea Ea's life.

$$\begin{array}{c|cc}
\text{Step} & P & Q \\ \hline
1 & \begin{array}{|c|}\hline 2 \\ \hline\end{array} & \begin{array}{|c|}\hline 1 \\ \hline\end{array} \\[1em]
2 & \begin{array}{|c|c|}\hline 2 & 5 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \\[1em]
3 & \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \\[1em]
4 & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \\[1em]
5 & \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}
\end{array}$$

Thus, we have $w \mapsto (P(w), Q(w))$, where the pair $(P(w), Q(w))$ is given by

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \qquad Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}.$$

For $w \in S_n$, we let $\mathrm{RS}(w) = (P(w), Q(w))$. Let $\mathrm{shape}(w)$ be the partition λ such that $P(w), Q(w) \in SYT(\lambda)$.

Fact 19.10. We have that $P(w^{-1}) = Q(w)$ and $Q(w^{-1}) = P(w)$.

19.3. **Jeu de Taquin and Promotion.** Start with some standard Young Tableau. Delete 1, and slide the smallest number below or right of the empty box into the empty box; repeat this process until the empty box is a corner. Place $n+1$ into the empty box and decrement all of the entries by 1. The sliding process is called *Jeu de Taquin*, and the resulting tableau is denoted by $\mathrm{Pro}(T)$.

Example 19.11. The tableau T, given by

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 9 \\ \hline 3 & 6 & 8 \\ \cline{1-3} 4 & 7 \\ \cline{1-2}\end{array},$$

is taken to

$$\begin{array}{|c|c|c|c|}\hline 1 & 4 & 7 & 8 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 3 & 6 \\ \cline{1-2}\end{array}.$$

More generally, *promotion* can be defined on the set $\mathcal{L}(P)$ of linear extensions of a finite poset.

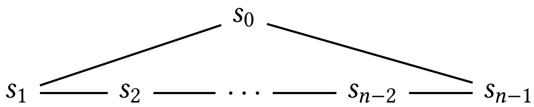
Example 19.12. Add example later.

One can check that

$$\mathrm{Pro} = \mathrm{BK}_{n-1} \circ \cdots \circ \mathrm{BK}_2 \circ \mathrm{BK}_1;$$

as an exercise, try this yourself on the example above.

Let $\partial_i = \mathrm{BK}_i \circ \cdots \circ \mathrm{BK}_2 \circ \mathrm{BK}_1$. Then $\mathrm{Pro} = \partial_{n-1}$. We define *evacuation* to be the map $\mathrm{Ev} : \mathcal{L}(P) \to \mathcal{L}(P)$ defined by $\mathrm{Ev} = \partial_1 \circ \partial_2 \circ \cdots \circ \partial_{n-3} \circ \partial_{n-2} \circ \partial_{n-1}$.



20. Thursday November 9

NO CLASS 11/21

Recall: P is an n-element poset. $\mathcal{L}(P)$ is the set of linear extensions of P.

$$\mathrm{Pro} = \mathrm{BK}_{n-1} \circ ... \circ \mathrm{BK}_2 \circ \mathrm{BK}_1$$

$$\partial_i + \mathrm{BK}_i \circ ... \circ \mathrm{BK}_2 \circ \mathrm{BK}_1$$

$$\mathrm{Ev} = \partial_1 \circ \partial_2 \circ ... \circ \partial_{n-1}$$

Example 20.1. picture insert pictu

20.0.1. *Symmetries in Robinson-Schensted.* $P(w^{-1}) = Q(w)$, $Q(w^{-1}) = P(w)$.

$$P(ww_0) = P(w)'$$

$$Q(ww_0) = \mathrm{Ev}(P(w))$$

$$P(w_0ww_0) = \mathrm{Ev}(P(w))$$

$$Q(w_0ww_0) = \mathrm{Ev}(Q(w))$$

Example 20.2. picture insert pictu

Theorem 20.3 (Stanley)**.** *The number of reduced words for the long element $w_0 \in S_n$ is equal to*

$$f^{\delta_{n-1}} = |\operatorname{SYT}(\delta_{n-1})|,$$

where $\delta_{n-1} = (n-1, n-2, ..., 3, 2, 1)$.

Note: $n = 4$. By the Hook-Length Formula, insert pictu

$$f^{\delta_{n-1}} = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}...(2n-3)^1}.$$

Theorem 20.4 (Edelman-Greene bijection)**.** *Let $N = \binom{n}{2}$. Let $T \in \operatorname{SYT}(\delta_{n-1})$. Number the corners of δ_{n-1} as $1, ..., n-1$ from bottom to top. Let γ_k be the number of the corner containing the entry N in $\mathrm{Pro}^k(T)$.*

Let $\mathrm{EG}(T) = s_{\gamma_0}s_{\gamma_1}...s_{\gamma_{N-1}}$.

Example 20.5. $n = 4$. $\mathrm{EG}(T) = s_2s_3s_1s_2s_3s_1$. insert pictu

Note that $\mathrm{EG}(\mathrm{Pro}(T)) = s_{\gamma_0}s_{\gamma_1}...s_{\gamma_N}$, $w_0 = s_{\gamma_0}s_{\gamma_1}...s_{\gamma_{N-1}} = s_{\gamma_1}s_{\gamma_2}...s_{\gamma_N}$. Then $s_{\gamma_1}...s_{\gamma_{N-1}} = s_{\gamma_0}w_0 = w_0s_{\gamma_N}$, so $s_{\gamma_N} = w_0s_{\gamma_0}w_0$. Thus, $\gamma_N = n - \gamma_0$. Similarly, $\gamma_{N+1} = n - \gamma_1$. In general, $\gamma_{N+k} = n - \gamma_k$ for all $k \geq 0$. Note: $n - \gamma_k$ is the corner containing N in $\mathrm{Pro}^k(T')$.

$$n - \gamma_k(T) = \gamma_k(T').$$

Since EG is a bijection, it follows that $\mathrm{Pro}^N(T) = T'$. So $\mathrm{Pro}^{2N}(T) = T$.

20.0.2. *Dynamical Algebraic Combinatorics.* Let X be a set of combinatorial objects. Suppose $f : X \to X$ is some interesting function. Study what happens when we iterate f.

Example 20.6. Pop on a lattice.

Example 20.7. Kreweras complementation on noncrossing partitions.

Example 20.8. Pro on $\mathcal{L}(P)$.

Theorem 20.9 (Edelman-Greene, 1987). *The order of* $\mathrm{Pro} : \mathrm{SYT}(\delta_{n-1}) \to \mathrm{SYT}(\delta_{n-1})$ *is* $2\binom{n}{2} = 2|\delta_{n-1}|$.

Theorem 20.10 (Schützenberger, 1977). *If P is a rectangle poset, then the order of* $\mathrm{Pro} : \mathcal{L}(P) \to \mathcal{L}(P)$ *is* $|P|$.

Theorem 20.11 (Haiman, 1992). *If P is a shifted double staircase or a shifted trapezoid, then* $\mathrm{Pro} : \mathcal{L}(P) \to \mathcal{L}(P)$ *has order* $|P|$.

Theorem 20.12 (Hopkins-Rubey, 2022). *IF P is a chain of V's, then the order of* $\mathrm{Pro} : \mathcal{L}(P) \to \mathcal{L}(P)$ *is* $2|P|$.

These are all the posets for which “nice" orbits under promotion are known to exist.

Definition 20.13. [Reiner-Stanton-White, 2004] Let X be a finite set. Suppose $f : X \to X$ is a bijection of order ω. Let $F(q) \in \mathbb{C}[q]$. We say $(X, f, F(q))$ exhibits the *cyclic sieving phenomenon* if for every $k \in \mathbb{Z}$, the number of elements of X fixed by f^k is

$$F(e^{2\pi ik/\omega}).$$

Example 20.14. $\mathrm{Pro} : \mathrm{SYT}(3,3) \to \mathrm{SYT}(3,3)$

Let $F(q) = (1 - q + q^2)(1 + q + q^2 + q^3 + q^4)$. $F(e^{2\pi i/6}) = 0$.

$$\begin{aligned} F(e^{2\pi i/6}) &= 0 \\ F(e^{2\pi i(2)/6}) &= 2 \\ F(e^{2\pi i(3)/6}) &= 3 \\ F(e^{2\pi i(4)/6}) &= 2 \\ F(e^{2\pi i(5)/6}) &= 0 \\ F(e^{2\pi i(6)/6}) &= 5 \end{aligned}$$

21. Tuesday November 14

Recall: that $[k]_q = (1-q^k)/(1-q) = 1+q+q^2+\cdots+q^{k-1}$ and that $[k]!_q = [k]_q[k-1]_q \cdots [1]_q$. Suppose X is a finite set, $f : X \to X$ is an invertible map of order ω, and $F(q) \in \mathbb{C}[q]$. We say $(X, f, F(q))$ exhibits the cyclic sieving phenomenon if for all $k \in \mathbb{Z}$, the number of fixed points of f^k is $F(e^{2\pi ik/\omega})$.

Theorem 21.1 (Rhoades, 2010). *Suppose $\lambda \in \pi(n)$ is a rectangular partition. The triple*

$$\left(\mathrm{SYT}(\lambda), \mathrm{Pro}, \frac{[n]!_q}{\prod_{b\in\lambda}[h_b]_q}\right)$$

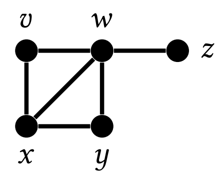
exhibits the cyclic sieving phenomenon.

21.1. **Toric Promotion.** Let $G = (V, E)$ be a graph with n vertices. A *labeling* of G is a bijection $V \to [n]$. Let Λ_G be the set of labelings of G. Given a labeling $\sigma \in \Lambda_G$, we obtain an acyclic orientation of G by directing each edge $\{u, v\}$ from u to v if $\sigma(u) < \sigma(v)$. This yields a poset $(V, \leq_\sigma)$ where $u \leq_\sigma v$ if there is a directed path from u to v. For $i \in [n-1]$, define $\mathrm{BK}_i : \Lambda_G \to \Lambda_G$ by

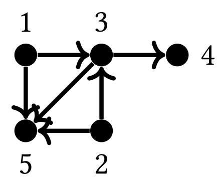
$$\mathrm{BK}_i(\sigma) = \begin{cases} (i\ i+1)\sigma & \text{if } \sigma^{-1}(i) \text{ and } \sigma^{-1}(i+1) \text{ are not adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 21.2. If $1 \leq i \leq n-1$, then σ and $\mathrm{BK}_i(\sigma)$ induce the same partial order on V. Also, σ and $\mathrm{BK}_i(\sigma)$ are both linear extensions of this poset and BK_i is the same as before.

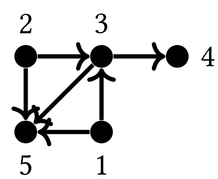
Example 21.3. Consider the graph G



with labeling σ given by



Note that we have drawn the associated acyclic orientation in the above. Now, it is not difficult to see that $\mathrm{BK}_1(\sigma)$ is given by



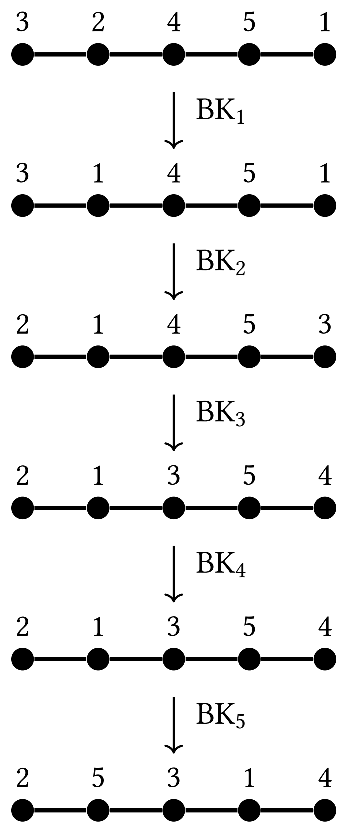
and we see that $(V, \leq_\sigma) = (V, \leq_{\mathrm{BK}_1(\sigma)})$.

So it makes sense to define $\mathrm{Pro} : \Lambda_G \to \Lambda_G$ by $\mathrm{Pro} = \mathrm{BK}_{n-1} \circ \cdots \circ \mathrm{BK}_2 \circ \mathrm{BK}_1$.

Idea: With the same setup as in the above, let G be labeled by $\mathbb{Z}/n\mathbb{Z}$ and define BK_n analogously. Define *toric promotion* to be the map $\mathrm{TPro} : \Lambda_G \to \Lambda_G$ given by

$$\mathrm{TPro} = \mathrm{BK}_n \circ \mathrm{BK}_{n-1} \circ \cdots \circ \mathrm{BK}_2 \circ \mathrm{BK}_1 .$$

Example 21.4. Consider the following labeled graph; we will apply toric promotion to it. Each step illustrates a Bender-Knuth toggle; the composition of all 5 steps is one application of TPro.



picture

Example 21.5. .

Theorem 21.6 (Defant, 2023)**.** *If G is a tree, then every orbit of* TPro *has size $n-1$.*

Idea: Let $\pi : [n] \longrightarrow \mathbb{Z}/n\mathbb{Z}$ be a bijection. Define *permutoric promotion* to be the map $\mathrm{TPro}_\pi : \Lambda_G \longrightarrow \Lambda_G$ given by $\mathrm{TPro}_\pi = \mathrm{BK}_{\pi(n)} \circ \mathrm{BK}_{\pi(n-1)} \circ \cdots \circ \mathrm{BK}_{\pi(1)}$. A *cyclic descent* of π^{-1} is an element $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\pi^{-1}(i) > \pi^{-1}(i+1)$. Let

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]!_q}{[b]!_q[a-b]!_q}.$$

Theorem 21.7 (Defant–Madhukara–Thomas, 2023+)**.** *Let $\pi : [n] \longrightarrow \mathbb{Z}/n\mathbb{Z}$ be a bijection. Let d be the number of cyclic descents of π^{-1}. The order of* $\mathrm{TPro}_\pi : \Lambda_{\mathrm{Path}_n} \longrightarrow \Lambda_{\mathrm{Path}_n}$ *is $n(n-d)$. Moreover,*

$$\left(\Lambda_{\mathrm{Path_n}}, \mathrm{TPro}_\pi, n(d-1)!(n-d-1)![n-d]_{q^d}\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)$$

exhibits the cyclic sieving property.

21.2. **Promotion Sorting.** Let $P = ([n], \leq_P)$ be a poset. Recall that $\mathrm{BK}_i : \mathcal{L}(P) \longrightarrow \mathcal{L}(P)$ is defined by

$$\mathrm{BK}_i(u) = \begin{cases} u & \text{if } u^{-1}(i) <_P u^{-1}(i+1); \\ s_i u & \text{otherwise.} \end{cases}$$

Define the *noninvertible Bender-Knuth toggle* $\tau_i : S_n \longrightarrow S_n$ by

$$\tau_i(u) = \begin{cases} u & \text{if } u^{-1}(i) <_P u^{-1}(i+1); \\ s_i u & \text{otherwise.} \end{cases}$$

Example 21.8. Add example later.

Define *extended promotion* to be the map $\text{Pro} = \tau_{n-1} \circ \cdots \circ \tau_2 \circ \tau_1$. Define *extended evacuation* to be the map $\text{Ev} = \partial_1 \circ \partial_2 \circ \cdots \circ \partial_{n-1}$, where $\partial_i = \tau_i \circ \cdots \circ \tau_2\tau_1$.

Theorem 21.9 (Defant–Kravitz, 2022). *We have* $\text{Pro}^{n-1}(S_n) = \mathcal{L}(P)$. *Also,* $\text{Ev}(S_n) = \mathcal{L}(P)$.

Define the *sorting time* of $w \in S_n$ to be the smallest $t \geq 0$ such that $\text{Pro}^t(w) \in \mathcal{L}(P)$. We say that w is *tangled* if it has sorting time $n-1$. Say w is quasitangled if it has sorting time $n-2$. Question: How many (quasi)tangled labelings does a given poset have? Answer: We don't know in general. Kravitz and Defant counted tangled labelings of posets that we call *inflated rooted trees.* Hodges counted quasitangled labelings of *inflated rooted trees with deflated leaves.* Hodges also found an algorithm to count labelings of P with sorting time $n-k-1$ when P is a rooted tree.

22. The Final Lecture

22.1. Bender–Knuth Billiards.

Recall that we have a poset $P = ([n], \leq_P)$ and $\tau_i : S_n \to S_n$

$$\tau_i(u) = \begin{cases} u & \text{if } u^{-1}(i) < u^{-1}(i+1); \\ s_i u & \text{otherwise.} \end{cases}$$

We have operators $\text{Pro} = \tau_{n-1} \circ \cdots \circ \tau_2 \circ \tau_1$ and $\text{Ev} = \tau_1 \circ (\tau_2 \circ \tau_1) \circ \cdots \circ (\tau_{n-1} \circ \cdots \circ \tau_2 \circ \tau_1)$.

Theorem 22.1 (Defant–Kravitz, 2022). *We have* $\text{Pro}^{n-1}(S_n) = \mathcal{L}(P)$ *and* $\text{Ev}(S_n) = \mathcal{L}(P)$.

Recall that we say $w \in S_n$ is *tangled* if $\text{Pro}^{n-2}(w) \notin \mathcal{L}(P)$.

Conjecture 22.2 (Defant–Kravitz, 2022). *The number of tangled labelings of P is at most* $(n-1)!$.

Let (W, S) be a Coxeter system. Let $S = \{s_i \mid i \in I\}$, where I is a finite index set. Let $\phi \subset V$ be the root system. Fix a convex set $\mathcal{L} \subset W$. Identify $w \in W$ with the region $w\mathbb{B}$ of the Coxeter arrangement $\mathcal{H}_W$. Say a hyperplane $H \in \mathcal{H}_W$ is a *window* if there exist elements of $\mathcal{L}$ on both sides of H. Say H is a one-way mirror if $\mathcal{L}$ lies on one side of H.

Example 22.3. Consider $W = S_n$. Recall that $\mathcal{L}$ is the set of linear extensions of some poset $P([n], \leq_P)$. The hyperplne $H_{ab} = \{x \in V^* \mid x_a = x_b\}$ is a window if and only if a and b are incomparable in P. If $a <_P b$, then $\mathcal{L}$ lies on the side of H_{ab} given by $x_a \leq x_b$, so H_{ab} is a one-way mirror. Suppose $u \in S_n$ and $1 \leq i \leq n-1$. Let $a = u^{-1}(i)$ and $b = u^{-1}(i+1)$. Then H_{ab} is the hyperplane separating u from $s_i u$. We have

$$\tau_i(u) = \begin{cases} s_i u & \text{if } H_{ab} \text{ is a window;} \\ s_i u & \text{if } \mathcal{L} \cup \{s_i u\} \text{ lies on one side of } H_{ab}; \\ u & \text{if } \mathcal{L} \cup \{u\} \text{ lies on one side of } H_{ab}. \end{cases}$$

Definition 22.4. (Barkley–Defant–Hodges–Kravitz–Lee) Suppose $u \in W$ and $i \in I$. Let $H \in \mathcal{H}_W$ be the hyperplane separating u from $s_i u$. Let

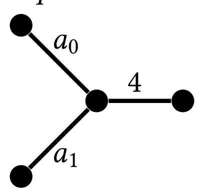
$$\tau_i(u) = \begin{cases} s_i u & \text{if } H \text{ is a window;} \\ s_i u & \text{if } \mathcal{L} \cup \{s_i u\} \text{ is on one side of } H; \\ u & \text{if } \mathcal{L} \cup \{u\} \text{ lies on one side of } H. \end{cases}$$

Fix an ordering $i_1, i_2, \ldots, i_n$ of I. This corresponds to choosing the standard Coxeter element $c = s_{i_n} s_{i_{n-1}} \cdots s_{i_1}$. Let $i_{j+n} = i_j$. Define $\mathrm{Pro}_c = \tau_{i_n} \circ \tau_{i_{n-1}} \circ \cdots \circ \tau_{i_1}$. Start at some $u_0 \in W$ and apply $\tau_{i_1}, \tau_{i_2}, \tau_{i_3}, \ldots$. This yields a sequence of elements $u_0, u_1, u_2, u_3, \ldots$, where $u_j = \tau_{i_j}(u_{j-1})$. We call this *Bender–Knuth billiards*, and we call $u_0, u_1, u_2, \ldots$ a billiards path. Does the billiards path always end up in $\mathcal{L}$ if $\mathcal{L}$ is finite?

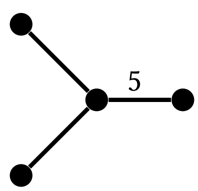
Definition 22.5. We say that $\mathcal{L}$ is *heavy* if every Coxeter element c and every starting point u_0, the billiards path eventually reaches $\mathcal{L}$. We say the Coxeter element c is *futuristic* if for every nonempty finite convex set $\mathcal{L}$ and every starting point u_0, the billiards path reaches $\mathcal{L}$. We say W is *futuristic* if every standard Coxeter element is futuristic (equivalently, every nonempty finite convex set is heavy). We say that W is *ancient* (or maybe *of yore*) if no Coxeter element is futuristic.

Theorem 22.6 (BDHKL). *The Coxeter system (W, S) is futuristic if any of the following hold:*
(1) *W is finite;*
(2) *$W = \widetilde{S}_n$;*
(3) *$W = \widetilde{C}_{n-1}$;*
(4) *$|S| \leq 3$;*
(5) *the Coxeter graph is complete;*
(6) *W is right-angled ($m(s,s') \in \{2, \infty\}$ for all $s, s' \in S$ with $s \neq s'$).*

Theorem 22.7. *The following Coxeter groups are ancient:*



where $a_0, a_1 \in \{3, 4, \ldots\}$;



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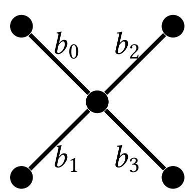

where $b_0, b_1, b_2, b_3 \in \{3, 4, \ldots\}$.

Likewise, we may define $\mathrm{Ev} = \tau_1 \circ (\tau_2 \circ \tau_1) \circ \cdots \circ (\tau_{n-1} \circ \cdots \circ \tau_2 \circ \tau_1)$. This is obtained from the reduced word $s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_{n-1} s_{n-2} \cdots s_2 s_1$ for w_0 by replacing each s_i by τ_i.

Theorem 22.8 (BDHKL). *Assume W is finite. Let $\mathcal{L} \subset W$ be a convex set. If $s_{i_N} \cdots s_{i_1}$ is a reduced word for w_0, then $(\tau_{i_N} \circ \cdots \circ \tau_{i_1})(W) = \mathcal{L}$.*